THE CLASSIFICATION OF RADIAL ENDS OF CONVEX REAL PROJECTIVE ORBIFOLDS

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Abstract. Real projective structures on n-orbifolds are useful in understanding the space of representations of discrete groups into $\mathsf{SL}(n+1,\mathbb{R})$. A recent work shows that many hyperbolic manifolds deform to manifolds with such structures not projectively equivalent to the original ones. The purpose of this paper is to understand the structure of ends of real projective n-dimensional orbifolds. In particular, we will study ones with the radial ends. These include hyperbolic manifolds with cusps and hyperideal ends. The main techniques are the theory of Fried and Goldman on affine manifolds and the work on Riemannian foliations by Molino, Carrière, and so on. For this, we will need to study the natural conditions on eigenvalues of holonomy representations of ends when these are manageably understandable. We will show that only the lens type ends or horospherical ends exist for irreducible properly convex real projective orbifolds under the conditions. We also discuss the ends completed by totally geodesic boundary orbifold and its duality to radial ends.

An *orbifold* is a topological space with charts modeling open sets by quotients of Euclidean open sets or half-open sets by finite group actions and compatibly patched with one another. The boundary of an orbifold is defined as the set of points with only half-open sets as models. Orbifolds are stratified by manifolds. Let $\mathcal{O}$ denote an n-dimensional orbifold with finitely many ends with end neighborhoods homeomorphic to a closed $(n-1)$-dimensional orbifold multiplied by an open interval. We will require that $\mathcal{O}$ has a compact suborbifold K so that $\mathcal{O} - K$ is a disjoint union of end neighborhoods homeomorphic to closed $(n-1)$-dimensional orbifolds multiplied by open intervals and hence $\partial\mathcal{O}$ is compact. This is a strong assumption; however, we note that the mathematicians have great difficulty understanding the topology of the ends of manifolds presently. (See [18] for an introduction to the geometric orbifold theory.)

An *orbifold covering map* is a map so that for a given modeling open set as above, the inverse image is a union of modeling open sets that are quotients as above. We say that an orbifold is a manifold if it has a subatlas of charts with trivial local groups. We will consider good orbifolds only. That is, it has a covering orbifold that is a simply connected manifold. In this case, the universal covering orbifold $\tilde{\mathcal{O}}$ is a manifold with an orbifold covering map $p_{\mathcal{O}} : \tilde{\mathcal{O}} \to \mathcal{O}$. We can even assume that it is very good; i.e., finitely covered by a manifold. The group of deck transformations will be denote

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by $\pi_1(\mathcal{O})$. We will denote it by $\mathbf{\Gamma}$ also throughout this paper. They act properly discontinuously on $\tilde{\mathcal{O}}$ but not necessarily freely.

By topological tameness, there exist only finitely many ends $E_1, \dots, E_m$, and each end has an end-neighborhood diffeomorphic to $\Sigma_{E_i} \times (0, 1)$. Let Σ_{E_i} here denote the orbifold type of the end E_i. The end neighborhoods of these types are said to be the *product type*.

Each end neighborhood U diffeomorphic to $\Sigma_E \times (0, 1)$ of an end E lifts to a connected open set $\tilde{U}$ in $\tilde{\mathcal{O}}$ where a subgroup of deck transformations $\mathbf{\Gamma}_{\tilde{U}}$ acts on $\tilde{U}$. $p_{\tilde{\mathcal{O}}}^{-1}(U)$ is a copy of $\tilde{U}$ under $\pi_1(\mathcal{O})$. $\tilde{U}$ is said to a *strict end neighborhood*. A sequence of strict end neighborhood in $\tilde{\mathcal{O}}$ is a system of neighborhoods

$$U_1 \supset U_2 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots .$$

A compatibility class of such systems is defined in Section 1.2.

The compatibility class of sequences of strict end neighborhoods is said to be an *end* in $\tilde{\mathcal{O}}$ here. (This is not strictly an "end of $\tilde{\mathcal{O}}$" in topological sense.) For an end E of $\tilde{\mathcal{O}}$, we denote by $\mathbf{\Gamma}_E$ the subgroup $\mathbf{\Gamma}_{\tilde{U}}$ where U and $\tilde{U}$ is as above. (Later, we identify E with a point of the closure of $\tilde{\mathcal{O}}$ in $\mathbb{RP}^n$.)

Generalizing, we will also say that a $\mathbf{\Gamma}_E$-invariant open subset containing an end neighborhood of E to an *end neighborhood*. Also, it is not an end-neighborhood of an end of $\tilde{\mathcal{O}}$. Note that these do not necessarily cover an end neighborhood in $\mathcal{O}$ but we will use this definition in this paper. Again an end is the compatibility class of sequence of end neighborhoods of these types. A strict end-neighborhood is one that does cover an end neighborhood in $\mathcal{O}$. Note that the notion of compatibility extends to the set of end neighborhoods not just strict ones and they define ends also.

We will denote also by $\pi_1(E)$ the subgroup $\mathbf{\Gamma}_E$ in $\pi_1(\tilde{\mathcal{O}})$. Note that the choice of the conjugacy class determines the end in $\tilde{\mathcal{O}}$.

Recall that the real projective space $\mathbb{RP}^n$ is defined as $\mathbb{R}^{n+1} - \{O\}$ under the quotient relation $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{O\}$. The general linear group $\mathsf{GL}(n+1, \mathbb{R})$ acts on $\mathbb{R}^{n+1}$ and $\mathsf{PGL}(n+1, \mathbb{R})$ acts faithfully on $\mathbb{RP}^n$. We will also use $\mathbb{S}^n$ as the double cover of $\mathbb{RP}^n$ and $\mathbf{Aut}(\mathbb{S}^n)$, isomorphic to the subgroup $\mathsf{SL}_{\pm}(n+1, \mathbb{R})$ of $\mathsf{GL}(n+1, \mathbb{R})$ of determinant ± 1, double-covers $\mathsf{PGL}(n+1, \mathbb{R})$ and acts a group of projective automorphisms of $\mathbb{S}^n$.

A *projective geodesic* is an arc developing into a straight line in $\mathbb{RP}^n$ or to a one-dimensional subspace of $\mathbb{S}^n$. An affine subspace A^n can be identified with the complement of a codimension-one subspace $\mathbb{RP}^{n-1}$ so that the geodesic structures are the same up to parameterizations. A *convex subset* of $\mathbb{RP}^n$ is a convex subset of an affine subspace in this paper. A *properly convex subset* of $\mathbb{RP}^n$ is a precompact convex subset of an affine subspace. $\mathbb{R}^n$ identifies with an open half-space in $\mathbb{S}^n$ defined by a linear function on $\mathbb{R}^{n+1}$.

We will consider an orbifold $\mathcal{O}$ with real projective structures: This can be expressed as having a pair $(\mathbf{dev}, h)$ where $\mathbf{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n$ is an immersion equivariant with respect to the homomorphism $h : \pi_1(\mathcal{O}) \to \mathsf{PGL}(n+1, \mathbb{R})$ where $\tilde{\mathcal{O}}$ is the universal cover and $\pi_1(\mathcal{O})$ is the group of deck transformations acting on $\tilde{\mathcal{O}}$. $(\mathbf{dev}, h)$ is only determined

up to an action of $\mathsf{PGL}(n+1,\mathbb{R})$ given by

$$g \circ (\mathbf{dev}, h(\cdot)) = (g \circ \mathbf{dev}, gh(\cdot)g^{-1}) \text{ for } g \in \mathsf{PGL}(n+1,\mathbb{R}).$$

We will use only one pair where $\mathbf{dev}$ is an embedding for this paper and hence identify $\tilde{\mathcal{O}}$ with its image.

We also have a lift $\mathbf{dev}' : \tilde{\mathcal{O}} \to \mathbb{S}^n$ and $h' : \pi_1(\mathcal{O}) \to \mathsf{SL}_\pm(n+1,\mathbb{R})$, which are also called developing maps and holonomy homomorphisms. The discussions below apply to $\mathbb{RP}^n$ and $\mathbb{S}^n$ equally. We will use this pair as $(\mathbf{dev}, h)$. Fixing $\mathbf{dev}$, we can identify $\tilde{\mathcal{O}}$ with $\mathbf{dev}(\tilde{\mathcal{O}})$ in $\mathbb{S}^n$. This identifies $\pi_1(\mathcal{O})$ with a group of projective automorphisms $\mathbf{\Gamma}$ in $\mathbf{Aut}(\mathbb{S}^n)$. This will be done in this paper.

Again an affine subspace in $\mathbb{S}^n$ is a lift of an affine space in $\mathbb{RP}^n$, which is the interior of an n-hemisphere. Convexity and proper convexity in $\mathbb{S}^n$ are defined in the same way as in $\mathbb{RP}^n$. We assume in this paper that $\mathcal{O}$ is a properly convex; i.e., $\tilde{\mathcal{O}}$ is a properly convex open domain. Hence, its closure is a compact n-ball. We will assume that our real projective orbifold $\mathcal{O}$ is a topologically tame orbifold and each end is *radial*. This means that each end has a neighborhood U, and each component $\tilde{U}$ of the inverse image $p_{\mathcal{O}}^{-1}(U)$, there exists a foliation by properly embedded projective geodesics ending at a common point $\mathbf{v}_{\tilde{U}} \in \mathbb{RP}^n$. We call such a point an *end vertex*.

A real projective orbifold with such a real projective structure with end neighborhoods radially foliated is said to have a *radial end structure*. Two such structures are *compatible* if the intersection of end neighborhoods are foliated with geodesics agreeing near the end vertices. We will assume that our orbifolds here have compatibility classes of radial end structures for all ends. That is, the topologically tameness *includes* this notion of the end structures in this paper. (We mainly do this since radial end structures for ends are not necessarily unique, while often it is unique, a posteriori facts that can be derived using the results here.)

Thus, each component of $p_{\mathcal{O}}^{-1}(U)$ has the same property, and the set of ends of $\tilde{\mathcal{O}}$ is in one-to-one correspondence with the set of end vertices in $\mathsf{Cl}(\tilde{\mathcal{O}})$. Thus, we denote by $\mathbf{v}_E$ also when E is the corresponding end of U.

The space of projective geodesics through $\mathbf{v}_E$ is a $(n-1)$-dimensional real projective space. We denote it by $\mathbb{RP}^{n-1}_{\mathbf{v}_E}$. Let Ω_E denote the space of radial lines from $\mathbf{v}_E$ in $\tilde{U}$ where two rays are regarded equivalent if they are identical near $\mathbf{v}_E$. Ω_E projects to a convex open domain in an affine space in $\mathbb{RP}^{n-1}_{\mathbf{v}_E}$ by the convexity of $\tilde{\mathcal{O}}$. Note that a subgroup $\mathbf{\Gamma}_E$ of $\mathbf{\Gamma}$ acts on as a projective automorphism group on $\mathbb{RP}^{n-1}_{\mathbf{v}_E}$. Thus, the quotient $\Omega_E/\mathbf{\Gamma}_E$ admits a real projective structure of one-dimension lower. We denote by Σ_E the real projective $(n-1)$-orbifold $\Omega_E/\mathbf{\Gamma}_E$. Since we can find a transversal orbifold Σ_E to the radial foliation in an end neighborhood for each end E' of $\mathcal{O}$, it lifts to a transversal surface Ω_E in $\tilde{U}$.

Define $\mathsf{bd}A$ for a subset A of $\mathbb{RP}^n$ to be the *topological boundary* in $\mathbb{RP}^n$ and define ∂A for a manifold or orbifold A to be the *manifold or orbifold boundary*. The closure $\mathsf{Cl}(A)$ of a subset A of $\mathbb{RP}^n$ is the topological closure in $\mathbb{RP}^n$.

In the following, all the sets are required to be inside an affine subspace.

- A subdomain K of $\mathbb{RP}^n$ is said to be *horospherical* if it is strictly convex and the boundary ∂K is diffeomorphic to $\mathbb{R}^{n-1}$ and $\mathsf{bd}K - \partial K$ is a single point.

- K is *lens-shaped* if it is a convex domain and ∂K is a disjoint union of two smoothly embedded $(n-1)$-cells not containing any straight segment in them.
- A *cone* is a domain in $\mathbb{RP}^n$ whose closure in $\mathbb{RP}^n$ has a point in the boundary, called an *end vertex* so that every other point has a segment contained in the domain with endpoint the cone point and itself.
- A *cone-over* a lens-shaped domain A is a convex submanifold and a cone that contains a lens-shaped domain A of the same dimension and is a union of segments from the end vertex $\notin A$ to points of A and the manifold boundary is one of the two boundary components of A. A *lens* is the lens-shaped domain A, not determined uniquely by the lens-cone itself.
- A *totally-geodesic subdomain* is a convex domain in a hyperspace. A *cone-over* a totally-geodesic domain A is a union of all segments with one end point a point x not in the hyperspace and the other in A.

An end of $\tilde{\mathcal{O}}$ is *horospherical* if it has a horospherical domain as an end neighborhood. It is *lens-shaped* or *totally geodesic cone-shaped* if it has an end neighborhood that is a lens-cone or a cone over a totally-geodesic domain.

We will later see that horospherical end neighborhoods are projectively diffeomorphic to end neighborhoods of hyperbolic orbifolds. Let E be an end and $\mathbf{\Gamma}_E$ the associated end holonomy group. If every subgroup of finite index of a group $\mathbf{\Gamma}_E \subset \mathbf{\Gamma}$ has a finite center, $\mathbf{\Gamma}_E$ is said to be a *virtual center-free group* or a *vcf-group*. An *admissible group* is a finite extension of a finite product of $\mathbb{Z}^{k-1} \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_k$ for trivial or infinite hyperbolic groups $\mathbf{\Gamma}_i$ in the sense of Gromov. (See Section 1.4 for details.) (For example, if our orbifold has a complete hyperbolic structure, then end fundamental groups are virtually free abelian.) If we understand the geodesic ergodicity properties of vcf-groups, we would be able to expand the class of groups of $\mathbf{\Gamma}_i$. However, we are not able to understand the properties yet. (We concentrate on orbifolds with end fundamental groups in this class.)

Let $\mathbf{\Gamma}$ be generated by finitely many elements $g_1, \dots, g_m$. The *conjugate word length* $\mathsf{cwl}(g)$ of $g \in \pi_1(E)$ is the minimum of the word length of the conjugates of g in $\pi_1(E)$. We will simply say this is the word length of g by an abuse of notation.

Let $\mathbf{v}_E$ be an end vertex of an end E. We assume that $\mathbf{\Gamma}_E$ is admissible and the associated real projective orbifold Σ_E is properly convex. $\mathbf{\Gamma}_E$ acts on a join $K := K_1 * \cdots * K_{l_0}$ where K_j is a properly convex domain in a projective sphere $\mathbb{S}^{i_j}$ of dimension $i_j \geq 0$ by Benoist [2]. $\mathbf{\Gamma}_E$ is virtually a direct product of $\mathbf{\Gamma}_j$ acting on K_j cocompactly and a center isomorphic to $\mathbb{Z}^{l_0-1}$ acting as a diagonalizable group fixing each point of K_j for $j = 1, \dots, l_0$. For convenience, we will identify $\mathbf{\Gamma}_E$ with $\mathbb{Z}^{l_0-1} \times \mathbf{\Gamma}_1 \times \cdots \mathbf{\Gamma}_{l_0}$ where $\mathbb{Z}^{l_0-1}$ denote the center by taking a finite index subgroup if necessary.

Let $\mathsf{length}_K(g)$ denote the infinum of $\{d_K(x, g(x)) | x \in K\}$, compatible with $\mathsf{cwl}(g)$. Thus, $\mathbf{\Gamma}_E$ restricts to a semisimple group $\mathbf{\Gamma}_j$ acting on K_j for some $j = 1, \dots, l_0$ and also contains the central abelain group $\mathbb{Z}^{l_0-1}$. The admissibility implies that $\mathbf{\Gamma}_j$ is a hyperbolic group. The end fundamental group $\mathbf{\Gamma}_E$ satisfies the *uniform middle-eigenvalue condition* if each irreducible factor $\mathbf{\Gamma}_i$ and $\mathbb{Z}^{l_0-1}$ satisfies for a uniform $C > 0$

independent of g

$$(1) \quad C^{-1}\mathrm{length}_K(g) \leq \log\left(\frac{\lambda_1(g)}{\lambda_{\mathbf{v}_E}(g)}\right) \leq C\mathrm{length}_K(g), g \in \mathbf{\Gamma}_i - \{\mathrm{I}\} \text{ or } g \in \mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$$

for the largest eigenvalue norm $\lambda_1(g)$ of g and the eigenvalue $\lambda_{\mathbf{v}_E}(g)$ of g at $\mathbf{v}_E$. (This condition is similar to ones studied by Guichard and Wienhard [35], and the results also seem similar. Our main tools to understand these questions are in Appendix A, and the author do not really know the precise relationship here.)

If we require only $\lambda_1(g) \geq \lambda_{\mathbf{v}_E}(g)$ for g in the center $\mathbb{Z}^{l_0-1}$, then we say that $\mathbf{\Gamma}_E$ satisfies the *weak uniform middle-eigenvalue conditions*. (The condition is needed since we need to study the limit representations as well in our later work.)

If Σ_E is not properly convex, then we will modify this definition in Section 5. Here $\mathbf{\Gamma}_E$ will act on a properly convex domain K of lower-dimension and we will apply the definition here.

It is not necessarily true that $\lambda_1(g) \leq \lambda_{\mathbf{v}_E}(g)$ a priori for $g \in \mathbf{\Gamma}_E$ at the moment. The condition is an open condition; and hence a "structurally stable one." (See Corollary 4.7.)

Furthermore, the dual of a radial end E is an end E' that compactifies to a totally geodesic boundary component. A necessary condition for the boundary component to have a convex lens-shaped neighborhood is this condition for the representation of $\pi_1(E')$ dual to $\pi_1(E)$. (See Section 3.4.) Also, if the end fundamental groups have trivial abelianizations as in many examples, this will hold. Hence, we consider this condition to be a natural one. (See Proposition 2.1.)

We say that $\mathcal{O}$ or Γ satisfies the *finite-essential-annulus condition* if the intersection of the infinitely many distinct end fundamental group is a trivial group.

An *ellipsoid* is a subset in an affine space defined as a zero locus of a positive definite quadratic polynomial in term of the affine coordinates. A projective conjugate of a parabolic subgroup $\mathcal{H}_{\mathbf{v}}$ of $\mathrm{SO}(i_0+1,1)$ acting on an i_0-dimensional ellipsoid with a point removed cocompactly is called an *i_0-dimensional cusp group*. If the horospherical neighborhood with the end vertex $\mathbf{v}$ has discrete cocompact subgroup in $\mathcal{H}_{\mathbf{v}}$, then we call the end to be of *cusp type*.

Our main result is:

Theorem 0.1. *Let $\mathcal{O}$ be the ends of properly convex tame real projective orbifold with radial ends and radial end structures. We assume that $\partial\mathcal{O}$ is compact. The fundamental group $\mathbf{\Gamma}$ is virtually irreducible or satisfies the finite-essential-annulus condition, and each end fundamental group is virtually isomorphic to a direct product of hyperbolic groups and infinite cyclic groups.*

- *Suppose that each end holonomy group satisfies the weakly uniform middle-eigenvalue condition. Then each end is either of lens type or cusp type or quasi-joined cusp type.*
- *Suppose that each end holonomy group satisfies the uniform middle-eigenvalue condition and the holonomy group of $\mathcal{O}$ is irreducible. Then each end is either of lens type or cusp type.*

We will explain the quasi-joined type in Section 5.2.

In Section 3, we define the dual domain Ω^* to Ω where the dual group Γ^* to Γ acts on. We show that a radial end corresponds to an end that can be completed by a totally geodesic orbifold of dimension $n-1$. Then the uniform middle-eigenvalue condition is shown to be equivalent to the boundary component to have a properly and strictly convex neighborhood in some ambient orbifold containing Ω^*/Γ^* and the boundary component. (See Section 3.4.) Hence, we think that the condition is a natural one.

If we can assume the uniform middle-eigenvalue condition for the ends, then the proof is simpler. But we prove a more general theorem with a little more work. See also Marquis for the end theory of 2-orbifolds...

A summary of the deformation spaces of real projective structures on orbifolds and surfaces is given in [18] and [11]. See also Marquis [45] for the end theory of 2-orbifolds. The deformation space of real projective structures on an orbifold loosely speaking is the space of isotopy equivalent real projective structures on a given orbitold.

To motivate why we think that Theorem 0.1 is important, we sketch some history: It was discovered by D. Cooper, D. Long, and M. Thistlethwaite [22], [23] that many closed hyperbolic 3-manifolds deform to projective 3-manifolds. Later S. Tillmann found an example of a 3-orbifold obtained from pasting sides of a single ideal hyperbolic tetrahedron admitting a complete cusped hyperbolic structure with a one-parameter family of real projective structure deformed from the hyperbolic one (see [13]). Also, Craig Hodgson, Gye-Seon Lee, and I found a few other examples: 3-dimensional ideal hyperbolic Coxeter orbifolds without edges of order 3 has at least 6-dimensional deformation spaces in [21].

Crampon-Marquis [25] and Cooper, Long, Tillmann [24] have done similar study with the finite volume condition. In this case, only possible ends are horospherical ones. The work here studies more general type ends and the proofs required stronger results but we have benefited from their work. We will see that there are examples where horospherical ends deform to lens-type ones and vice versa (see also Example 2.3.)

Our main aim is to understand these phenomena theoretically. It became clear from our attempt in [13] that we need to understand and classify the types of ends of the relevant convex real projective orbifolds. We will start with the simplest ones: radial type ones. But as Mike Davis observed, there are many other types such as ones preserving subspaces of dimension greater than equal to 0. We will not present any of them here; however, its seems very likely that many techniques here will be applicable.

In [13], we show that the deformation spaces of real projective structures on orbifolds are locally homeomorphic to the spaces of conjugacy classes of representations their fundamental group where both spaces are restricted by some conditions.

It remains how to see for which of these types of real orbifolds, nontrivial deformations exist or not for a given example such as a complete hyperbolic manifolds and how to compute the deformation space. However, from Theorem 1 in [21] with Coxeter orbifolds, we know that a complete hyperbolic Coxeter orbifold always deforms nontrivially. (Note that we can extend Theorem 1 to include all the cases where the edge orders are ≥ 3 from Remark 4 of [21].) We conjecture that maybe these types of real projective orbifolds with radial ends might be very flexible. Of course, we have no real

analytical or algebraic means to understand these phenomena; hence, the complete understanding is left to some future work.

In Section 1, we go over basic definitions. We discuss ends of orbifolds, convexity, the Benoist theory on convex divisible actions, and so on.

In Section 2, we discuss examples of ends; horospherical ones, totally geodesic ones, and bendings of ends.

In Section 3, we start to study the end theory. First, we discuss the holonomy representation spaces. Tubular actions and the dual theory of affine actions are discussed. We show that distanced actions and asymptotically nice actions are dual.

We show that the dual of a radial end is a totally geodesic or horospherical. We show that uniform middle-eigenvalue condition is natural in the sense that the condition is equivalent to the existence of a strictly convex neighborhood.

We discuss about horospherical ends. First, they are complete ends and have holonomy matrices with unit norm eigenvalues only and their end fundamental groups are virtually nilpotent. Conversely, a complete end in a properly convex orbifold has to be of cusp-type.

We discuss the properties of lens-shaped ends. We show that if the holonomy is irreducible, the lens shaped ends have concave neighborhoods. If the lens-shaped end is reducible, then it can be made into a totally-geodesic conical end, which is a surprising result in the author's opinion. Finally, we show that the lens shaped property is a stable property under the change of holonomy representations.

In Section 4, we introduce the weakly uniform middle-eigenvalue conditions for end holonomy groups. We show that the uniform middle-eigenvalue condition of an end is equivalent to the lens-shaped property of the end.

In Section 5, we discuss the ends that are convex but not properly convex in the transverse sense. First, we show that the end holonomy group for an end E will have an exact sequence

$$1 \rightarrow N \rightarrow h(\pi_1(E)) \xrightarrow{\pi_K} N_K \rightarrow 1$$

where N_K is in the projective automorphism group $\mathbf{Aut}(K)$ of a properly convex open set K and N is the normal subgroup mapped to the trivial automorphism of K. We show that Σ_E is foliated by complete affine spaces of dimension ≥ 1.

First, we discuss the case when N_K is a discrete. Here, N is virtually abelian and is conjugate to a discrete cocompact subgroup of a cusp group. We introduce the example of joining of horospherical and lens type ends. By computations involving the normalization conditions, we show that the above exact sequence is virtually split and we can suprisingly show that the ends are of join or quasi-join types.

Next, we discuss the case when N_K is not discrete. Here, there is a foliation by complete affine spaces as above. The leaf closures are submanifolds V_l by the theory of Molino [48] on Riemannian foliations. We use some estimate to show that each leaf is of polynomial growth. This shows that the identity component of the closure of N_K is abelian and $\pi_1(V_l)$ is solvable using the work of Carrère [9]. One can then take the syndetic closure to obtain a bigger group that act transitively on each leaf. Then we find a normal cusp group acting on each leaf transitively. Then we show that the end also splits virtually.

Finally for both of these cases, we show that the orbifold has to be reducible by considering the limit actions of some elements in the joined ends. This proves that the joined end does not exist, proving Theorem 0.1.

In Appendix A, we show that the affine action of irreducible group Γ acting cocompactly on a convex domain Ω in the boundary of the affine space is asymptotically nice if Γ satisfies the uniform middle-eigenvalue condition. This was needed in Section 5.

We thank David Fried for helping me understand the issues with distanced nature of the tubular actions and Yves Carrière with the general approach to study the indiscrete cases for nonproperly convex ends. The basic Lie group approach of Riemannian foliations was a key idea here as well as the theory of Fried on distal groups. We thank Yves Benoist with some initial discussions on this topic, which were very helpful for Section 3.1 and thank Bill Goldman and Francois Labourie for discussions related to Appendix A.3 inspired by their paper [32]. We thank Daryle Cooper and Stephan Tillmann for explaining their work and help and we also thank Mickaël Crampon and Ludovic Marquis also. Their work obviously was influential here. The study was begun with a conversation with Tillmann in "Manifolds at Melbourne 2006" and I began to work on this seriously from my sabbatical year at Univ. Melbourne from 2008. We also thank Craig Hodgson and Gye-Seon Lee for working with me with many examples and their insights. The idea of radial ends comes from the cooperation with them.

1. Preliminary

In this paper, we will be using the smooth category: that is, we will be using smooth maps and smooth charts and so on. We explain the material in the introduction again.

1.1. Real projective structures.

1.1. Real projective structures. Denote by $\mathbb{R}_+ = \{r \in \mathbb{R} | r > 0\}$. Let O denote the origin of any vector space here. Given a vector space V, we denote by $\mathcal{P}(V)$ the projective space $(V - \{O\})/\sim$ where $\vec{v} \sim \vec{w}$ iff $\vec{v} = s\vec{w}$ for $s \in \mathbb{R} - \{0\}$ and we denote by $\mathcal{S}(V)$ the sphere $(V - \{O\})/\sim$ where $\vec{v} \sim \vec{w}$ for $s \in \mathbb{R}_+$. Note we denote $\mathbb{RP}^n = \mathcal{P}(\mathbb{R}^{n+1})$ and $\mathbb{S}^n = \mathcal{S}(\mathbb{R}^{n+1})$. A *subspace* of $\mathcal{P}(V)$ or $\mathcal{S}(V)$ is the image of a subspace in V with O removed. Given any linear isomorphism $f : V \to W$, we denote by $\mathcal{P}(f)$ the induced projective isomorphism $\mathcal{P}(V) \to \mathcal{P}(W)$ and $\mathcal{S}(f)$ the induced map $\mathcal{S}(V) \to \mathcal{S}(W)$.

An affine space A^n is a vector space $\mathbb{R}^n$ with translation group added as the automorphism group $\mathbf{Aff}(A^n)$. Here affine geodesics are defined to be the straight lines.

The complement of a codimension-one subspace in $\mathbb{RP}^n$ can be considered an affine space $\mathbb{R}^n$ by correspondence

$$[1, x_1, \dots, x_n] \to (x_1, \dots, x_n)$$

for a coordinate system where the codimension-one subspace is given by $x_0 = 0$. The group of projective automorphism $\mathbf{Aff}(A^n)$ acting on A^n is identical with the group of affine transformations of form

$$\vec{x} \mapsto A\vec{x} + \vec{b}$$

for a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$. The projective geodesics and the affine geodesics agree up to parametrizations.

Note that we can double-cover $\mathbb{RP}^n$ by $\mathbb{S}^n$ the unit sphere in $\mathbb{R}^{n+1}$ and this induces a real projective structure on $\mathbb{S}^n$. We call this the real projective sphere. The antipodal map $\mathcal{A} : \mathbb{S}^n \to \mathbb{S}^n$ given by $[\vec{v}] \to [-\vec{v}]$ for $\vec{v} \in \mathbb{R}^{n+1} - \{O\}$ which generates the covering automorphism group of $\mathbb{S}^n \to \mathbb{RP}^n$. The group $\mathbf{Aut}(\mathbb{S}^n)$ of projective automorphisms of $\mathbb{S}^n$ is isomorphic to $\mathsf{SL}_\pm(n+1, \mathbb{R})$.

Given a projective structure where $\mathbf{dev} : \tilde{\mathcal{O}} \to \mathbb{RP}^n$ is an embedding to a properly convex open subset as in this paper, there is a lift and an embedding $\mathbf{dev}' : \tilde{\mathcal{O}} \to \mathbb{S}^n$ to an open domain D without any pair of antipodal points. D is determined up to the antipodal map

We will identify $\tilde{\mathcal{O}}$ with D or $\mathcal{A}(D)$ and $\pi_1(\mathcal{O})$ or $\mathbf{\Gamma}$ lifts to a subgroup of $\mathsf{SL}_\pm(n+1, \mathbb{R})$ acting faithfully and discretely on $\tilde{\mathcal{O}}$. Thus, we also define the end vertices of ends of $\tilde{\mathcal{O}}$ as points in the boundary of $\tilde{\mathcal{O}}$ in $\mathbb{S}^n$ from now on.

We will think of $\tilde{\mathcal{O}}$ as a convex domain in $\mathbb{S}^n$ and $\mathbf{\Gamma}$ a subgroup of $\mathsf{SL}_\pm(n+1, \mathbb{R})$ acting on $\mathbb{S}^n$.

1.2. **Ends.** Suppose that $\mathcal{O}$ is a topologically tame properly convex real projective orbifold with radial ends and a universal cover $\tilde{\mathcal{O}}$ with compact $\partial\mathcal{O}$ and radial end structures. (This will be the universal assumption for this paper.) We will drop the term "topologically tame" everywhere except for theorems.

Consider a sequence of open sets $U_1, U_2, ...$ so that $U_i \supset U_{i+1}$ where each U_i is a component of the complement of a compact subset in $\mathcal{O}$ and given each compact set K in $\mathcal{O}$, $U_i \cap K \neq \emptyset$ for only finitely many i. Such a sequence is said to be an *end neighborhood system*. Two such sequences $\{U_1, U_2, ...\}$ and $\{U'_1, U'_2, ...\}$ are equivalent if for each U_i we find k so that $U'_j \subset U_i$ for $j > k$ and conversely for each U'_i we find k' such that $U_j \subset U'_i$ for $j > k'$. An equivalence class of end neighborhoods is said to be an *end* of $\mathcal{O}$. A *neighborhood* of an end is one of the open set in the sequence in the equivalence class of the end.

A *radial end-neighborhood system* of $\mathcal{O}$ is the union of end neighborhoods for all ends disjoint from one another where each end neighborhood is of product type and is radially foliated compatibly with the product structure.

A *ray* is a convex geodesic segment with two end points; that is, it is convex and can contain a pair of antipodal points as the pair of end points only.

Given a component of such a system, we obtain that the inverse image is a disjoint union of connected open sets where we have the subgroup acting on one of them U denoted by $\mathbf{\Gamma}_U$ so that $U/\mathbf{\Gamma}_U$ is homeomorphic to the product end neighborhood. U is also foliated by radial rays ending at a common vertex $\mathbf{v}_U$. Note that any other component U' is of form $\gamma(U)$ for $\gamma \in \mathbf{\Gamma} - \mathbf{\Gamma}_U$ and $\mathbf{\Gamma}_{U'} = \gamma\mathbf{\Gamma}_U\gamma^{-1}$ and $\mathbf{v}_{U'} = \gamma(\mathbf{v}_U)$.

By an abuse of terminology, an open set U' containing U as above and $U'/\mathbf{\Gamma}_U$ is homeomorphic to $U/\mathbf{\Gamma}_U$ and foliated by radial rays ending at $\mathbf{v}_U$ will be called an *end neighborhood* of the *end vertex* $\mathbf{v}_U$. (One that does cover an end neighborhood of $\mathcal{O}$ is called a *strict* end neighborhood.) U' is not required to cover an open set in $\mathcal{O}$. Here, U' may be not be a real neighborhood in topological sense as in the cases of horospherical ends. We call $\mathbf{\Gamma}_U$ the end fundamental group. Up to the $\mathbf{\Gamma}$-action, there are only finitely many end vertices and end fundamental groups. For an end E, $\mathbf{\Gamma}_U$ is

well-defined up to conjugation by $\mathbf{\Gamma}$ and we denote it by $\mathbf{\Gamma}_E$ often for suitable choice of U. Its conjugacy class is more appropriately denoted $\mathbf{\Gamma}_E$.

Lemma 1.1. *Suppose that $\mathcal{O}$ is a properly convex real projective orbifold with radial ends and a universal cover $\tilde{\mathcal{O}}$ with $\partial\mathcal{O}$ compact. Let U be an end neighborhood. Let $\tilde{U}$ be the inverse image of the union U of mutually disjoint end neighborhoods. For a given component U_1 of $\tilde{U}$, we have if $\gamma(U_1) \cap U_1 \neq \emptyset$, then $\gamma(U_1) = U_1$ and γ lies in the fundamental group $\mathbf{\Gamma}_{E'}$ of the end E' associated with U_1.*

Proof. This follow since U_1 covers an end neighborhood. □

Lemma 1.2. *Let U be a radial end neighborhood of an end vertex $\mathbf{v}_E$ with $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ meeting each ray uniquely. Suppose that the boundary $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ of U maps into an end neighborhood of $\mathcal{O}$ under the covering map or equivalently $\mathrm{Cl}(U) \cap \tilde{\mathcal{O}}$ maps into the end neighborhood. Then $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ covers a compact hypersurface homotopy equivalent to the end Σ_E and its end neighborhood and $p_{\mathcal{O}}(U)$ is homeomorphic to $\Sigma_E \times \mathbb{R}$.*

Proof. Let V' be the end neighborhood of $\mathcal{O}$ that $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ or $\mathrm{Cl}(U) \cap \tilde{\mathcal{O}})$ maps into. Then for a component of the inverse image V of V', we have $U \subset V'$. Since $\mathbf{\Gamma}_E$ is precisely the set of deck-transformations acting on V, it follows that U covers $p_{\mathcal{O}}(U)$ in V' with the deck transformation group $\mathbf{\Gamma}_E$. Also, $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ covers the boundary of $p_{\mathcal{O}}(U)$ in V', and hence is a compact hypersurface. Since V is homeomorphic to $\Sigma_E \times \mathbb{R}$, the result follows. □

Proposition 1.3. *Suppose that $\mathcal{O}$ is a properly convex real projective orbifold with radial ends, and its developing map sends its universal cover $\tilde{\mathcal{O}}$ to a convex domain. Let U' be an end neighborhood in $\mathcal{O}$. Let $\tilde{U}$ be $p_{\mathcal{O}}^{-1}(U')$ as above with E' the end in $\tilde{\mathcal{O}}$ associated with a component U of $\tilde{U}$. Then the closure of each component of $\tilde{U}$ contains the endpoints $\mathbf{v}_{E'}$ of the leaf of radial foliation in $\tilde{U}$ lifted from U, and there exists a unique one for each component U_1 of $\tilde{U}$ associated with end E' of $\tilde{\mathcal{O}}$ and the subgroup of $h(\pi_1(\mathcal{O}))$ acting on it or fixing the end vertex $\mathbf{v}_E$ is precisely in the class $\mathbf{\Gamma}_{E'}$.*

We will show later that the end neighborhood can be chosen to be properly convex by taking convex hull of a well-chosen end neighborhood. However, there is no guarantee that the images of convex ones are disjoint.

1.3. **Convexity and convex domains.** The results here are originally due to Kuiper, Koszul, and so on. An affine manifold is *convex* if every path can be homotoped to an affine geodesic with endpoints fixed. A complete real line in $\mathbb{RP}^n$ is a 1-dimensional subspace of an affine space A^n. A convex affine manifold is *properly convex* if there is no affine map from $\mathbb{R}$ into it, i.e., there is no complete affine line in its universal cover.

A *parallel end* is an end with an end neighborhood U in the universal cover so that there exists a vector $\vec{v}$ so that $U + t\vec{v} \subset V$ for all $t > 0$.

Proposition 1.4 (Vey)**.** *A tame affine manifold with nonempty parallel end is convex if and only if the developing map sends the universal cover to a convex open domain in $\mathbb{R}^n$. A tame affine manifold with nonempty parallel end is properly convex if and only if the developing map sends the universal cover to a properly convex open domain in $\mathbb{R}^n$.*

Proof. The first part is Theorem 8.1 of Shima [50]. The second part is Theorem 8.3 of [50] since the hyperbolicity there is equivalent to proper convexity. (See Kobayashi [39].)

□

Lemma 1.5. *A properly convex subset of an affine subspace is a convex subset of a compact subset of an affine subspace.*

Proof. The proof is similar to the 2-dimensional situation in Section 1.3 of [15]. □

A complete real line in $\mathbb{RP}^n$ is a 1-dimensional subspace of $\mathbb{RP}^n$ with one point removed. That is, it is the intersection of a 1-dimensional subspace by an affine space. A *convex* projective geodesic is a projective geodesic in a real projective manifold which lifts to a projective geodesic, the image of whose composition with a developing map does not contain a complete real line. A real projective manifold is *convex* if every path can be homotoped to a convex projective geodesic with endpoints fixed. It is *properly convex* if there is no projective map from the complete real line $\mathbb{R}$ to it.

In the double cover $\mathbb{S}^n$ of $\mathbb{RP}^n$, an affine space A^n is the interior of a hemisphere. A domain in $\mathbb{RP}^n$ or $\mathbb{S}^n$ is *convex* if it lies in some affine subspace and satisfies the convexity property above. Note that a convex domain in $\mathbb{RP}^n$ lifts to ones in $\mathbb{S}^n$ up to the antipodal map $\mathcal{A}$ and a convex domain in $\mathbb{S}^n$ that does not contain an antipodal pair map to one in $\mathbb{RP}^n$ homeomorphic. (Actually from now on, we will only be interested in convex domains in $\mathbb{S}^n$.)

Proposition 1.6.
- *A tame real projective orbifold with nonempty radial end is convex if and only if the developing map sends the universal cover to a convex domain in $\mathbb{RP}^n$ or $\mathbb{S}^n$.*
- *A tame real projective orbifold with nonempty radial end is properly convex if and only if the developing map sends the universal cover to a properly convex open domain in a compact domain in an affine patch of $\mathbb{RP}^n$.*
- *If a tame convex real projective orbifold with nonempty radial end is not properly convex, then its holonomy is virtually reducible.*

Proof. The first part follows by affine suspension and Proposition 1.4. For the second part, the affine suspension has a developing image to a properly convex subset of an affine subspace A^n by Lemma 1.5. For the final item, a convex subset of $\mathbb{RP}^n$ is a convex subset of an affine patch A^n, isomorphic to an affine space. A convex subset of A^n that has a complete affine line must contain a maximal complete affine subspace. Two such complete maximal affine subspaces do not intersect since otherwise there is a larger complete affine subspace of higher dimension by convexity. We showed in [10] that the maximal complete affine subspaces are parallel. This implies that the boundary of the affine subspaces is a lower dimensional subspace. These subspaces are preserved under the group action. □

1.4. **The Benoist theory.** In 1990s Benoist more or less completed the theory of the divisible action as started by Benzecri, Vinberg, Koszul, Vey, and so on in series of papers [1, 2, 3, 4, 5, 6]. The comprehensive theory will aid us much in this paper.

See [3] for the proofs of the following propositions.

Proposition 1.7 (Benoist)**.** *Suppose that a discrete subgroup Γ of $\mathsf{SL}_\pm(n,\mathbb{R})$ acts on a properly convex $(n-1)$-dimensional open domain Ω in $\mathbb{S}^{n-1}$ so that Ω/Γ is compact. Then the following statements are equivalent.*

- *Every subgroup of finite index of Γ has a finite center.*
- *Every subgroup of finite index of Γ has a trivial center.*
- *Every subgroup of finite index of Γ is irreducible in $\mathsf{SL}_\pm(n,\mathbb{R})$. That is, Γ is strongly irreducible.*
- *The Zariski closure of Γ is semisimple.*
- *Γ does not contain a normal infinite nilpotent subgroup.*
- *Γ does not contain a normal infinite abelian subgroup.*

The group with properties above is said to be the group with a *trivial virtual center.*

Theorem 1.8 (Benoist)**.** *Let $n-1 \geq 1$. Let Γ be a discrete subgroup of $\mathsf{SL}_\pm(n,\mathbb{R})$ with a trivial virtual center. Suppose that a discrete subgroup Γ of $\mathsf{SL}_\pm(n,\mathbb{R})$ acts on a properly convex $(n-1)$-dimensional open domain $\Omega \subset \mathbb{S}^{n-1}$ so that Ω/Γ is compact. Then every representation of a component of $\mathsf{Hom}(\Gamma, \mathsf{SL}_\pm(n,\mathbb{R}))$ containing the inclusion representation also acts on a properly convex $(n-1)$-dimensional open domain cocompactly.*

We call the group such as above theorem a *vcf-group*. By above Proposition 1.7, we see that every representation of the group acts irreducibly.

The decompositions below will be called *Vey decompositions*.

Proposition 1.9 (Benoist [2])**.** *Assume $n \geq 2$. Let Σ be a closed $(n-1)$-dimensional properly convex projective orbifold and let Ω denote its universal cover. Then*

- *Ω is projectively diffeomorphic to a join $K_1 * \cdots * K_{l_0}$ where K_i is a properly convex open domain of dimension $n_i \geq 0$ in the subspace $\mathbb{S}^{n_i}$ in $\mathbb{S}^n$ corresponding to a convex cone $C_i \subset \mathbb{R}^{n_i+1}$.*
- *Ω is the image of $C_1 \oplus \cdots \oplus C_r$.*
- *The fundamental group $\pi_1(\Sigma)$ is virtually isomorphic to $\mathbb{Z}^{l_0-1} \times \mathbf{\Gamma}_1 \times \cdots \mathbf{\Gamma}_{l_0}$ for $l_0 - 1 + \sum n_i = n$.*
- *Each $\mathbf{\Gamma}_j$ acts on K_j cocompactly and the Zariski closure is*
$$\mathsf{SL}(n_i+1,\mathbb{R}), \mathsf{SL}_\pm(n_i+1), \mathsf{SO}(n_i,1) \text{ or } \mathsf{O}(n_i,1)$$
and acts trivially on K_m for $m \neq j$.
- *The subgroup corresponding to $\mathbb{Z}^{l_0-1}$ acts trivially on each K_j*

Proof. The first three items and the last one are from Theorem 1.1. in [2].

Let $\hat{h} : \pi_1(\Sigma) \to \mathsf{SL}_\pm(n,\mathbb{R})$ be the homomorphism associated with E. The first part of the fourth item is also from Theorem 1.1 of [2]. By Theorem 1.1 of [3], the Zariski closure of $\hat{h}(\pi_1(E))$ is virtually a product $\mathbb{R}^{l_0-1} \times G_1 \times \cdots \times G_{l_0}$ and G_j, $j = 1, \dots, l_0$, is an irreducible reductive Lie subgroup of $\mathsf{SL}_\pm(V_i)$. Suppose $\mathbf{\Gamma}_i$ acts nontrivially on C_k for $k \neq i$. Then elements of Zariski closures Z_k^i of their images commute in G_k and G_k is the centralizer of products of subgroups Z_k^is. Since G_k is irreducible linear algebraic subgroup, this is absurd. (We were helped by Benoist in this argument.)

□

Note here K_i could be a point. For some s, $1 \leq s \leq r$, we could obtain a decomposition where each K_i for $i \geq s$ has dimension ≥ 2 and $\mathbf{\Gamma}_i$ is a hyperbolic group. Then $\mathbf{\Gamma}$ is virtually a product of hyperbolic groups and an abelian group that is the center of the group. We will restrict our ends to be in this situation.

1.5. **Definitions associated with ends.** We will use:

Definition 1.10. Let $\mathcal{O}$ denote a convex real projective n-orbifold with radial ends with the universal cover $\tilde{\mathcal{O}} \subset \mathbb{S}^n$ and the group of deck transformation $\mathbf{\Gamma}$ acting on $\tilde{\mathcal{O}}$ projectively. Let $\mathbf{v}_E$ be the end vertex in $\mathbb{S}^n$ corresponding to an end E of $\tilde{\mathcal{O}}$. Let us denote by $\mathbb{RP}^{n-1}_{\mathbf{v}_E}$ the space of lines through $\mathbf{v}_E$. We need the sphere $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ of rays at $\mathbf{v}_E$ the double cover of $\mathbb{RP}^{n-1}_{\mathbf{v}_E}$. The subgroup of projective automorphisms $\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbf{v}_E}$ of $\mathbb{S}^n$ fixing $\mathbf{v}_E$ acts on it as projectivised to $\mathsf{SL}_\pm(n,\mathbb{R})$ acting on $\mathbb{S}^{n-1}$.

We can associate an end E of $\mathcal{O}$ with any end vertex $\mathbf{v}_E \in \mathbb{S}^n$. Thus, there are only finitely many orbit types for end vertices under $\mathbf{\Gamma}$. Sometimes, we denote $\mathbf{\Gamma}_{\mathbf{v}_E}$ by $\pi_1(E)$ also and is said to be the *end fundamental group* of E. Two rays from $\mathbf{v}_E$ are equivalent if they agree in a neighborhood of $\mathbf{v}_E$. Given an end E corresponding to $\mathbf{v}_E$, we denote by $R_{\mathbf{v}_E}(E)$ the space of equivalence classes of rays from $\mathbf{v}_E$ in $\tilde{\mathcal{O}}$. This is a convex domain since $\tilde{\mathcal{O}}$ is. Also, for a subset K of U_1, we denote by $R_{\mathbf{v}_E}(K)$, the space of rays from $\mathbf{v}_E$ ending at K, which is a convex set provided K is. We have $R_{\mathbf{v}_E}(\tilde{\mathcal{O}}), R_{\mathbf{v}_E}(K) \subset \mathbb{S}^{n-1}_{\mathbf{v}_E}$.

Note that the same notation is sometimes used for ends E of $\mathcal{O}$ itself.

A *properly convex end* is an end with a properly convex end $R_{\mathbf{v}_E}(\tilde{\mathcal{O}})$. A *complete end* is an end with $R_{\mathbf{v}_E}(\tilde{\mathcal{O}})$ a complete affine space.

Proposition 1.11. *Let $\mathcal{O}$ be a convex real projective orbifold with radial ends. Let E be an end of $\tilde{\mathcal{O}}$. If $\mathcal{O}$ is convex, $R_{\mathbf{v}_E}(E)$ is also a convex domain in an affine subspace of $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ for the associated vertex $\mathbf{v}_E$ for E. The group $h(\pi_1(E))$ induces a group $\hat{h}(\pi_1(E))$ of projective transformations of $\mathbb{S}^{n-1}$ acting on $R_{\mathbf{v}_E}(E)$ and $R_{\mathbf{v}_E}(E)/\hat{h}(\pi_1(E))$ is diffeomorphic to the end orbifold Σ_E and has the induced projective structure of E.*

Proof. Straightforward. $\square$

We remark also that for the orbifold with admissible ends and the infinite-index end fundamental group condition, the end vertices are infinitely many for each equivalence class of vertices.

2. Examples of ends

We will present some examples here. They will be fully justified later.

2.0.1. *Examples.* From hyperbolic manifolds, we can obtain some examples of ends. Let M be a complete hyperbolic manifolds with cusps. M is a quotient space of the interior Ω of a conic in $\mathbb{RP}^n$ or $\mathbb{S}^n$. Then the horoballs form the horospherical ends.

Suppose that M has totally geodesic embedded surfaces $S_1, .., S_m$ homotopic to the ends. Then $\pi_1(S_i)$ fixes a point outside the conic, and S_i acts on a lens-shaped domain that is an ϵ-neighborhood of S_i in $\Omega/\pi_1(M)$. Hence, we can add the cone over the lens-shaped domain to M to obtain the examples of real projective manifolds with radial

ends. (This is sometimes called the *hyperideal extension* of the hyperbolic manifolds as real projective manifolds.)

Proposition 2.1. *Suppose that M is a properly convex real projective orbifold with radial ends. Suppose that each end fundamental group is generated by the homotopy classes of closed curves about singularities or has the holonomy fixing the end vertex with eigenvalues 1. If an end has a compact totally geodesic properly convex hyperspace in an end neighborhood, then the end is of lens-type.*

Proof. Let $\tilde{M}$ be the universal cover of M in $\mathbb{S}^n$. Let E be an end of $\tilde{M}$ with a compact totally geodesic hyperspace Σ in an end neighborhood. Then there is an end neighborhood U containing the universal cover $\tilde{\Sigma}$ of Σ.

If each end fundamental group is generated by closed curves about singularities, then since the singularities are of finite order, the eigenvalues of the generators corresponding to the end vertex equal 1 and hence every element of the end fundamental group has 1 as the eigenvalue there. Now assume that the holonomy of the elements of the end fundamental group, fixes the end vertex with eigenvalues equal to 1.

Then U can be chosen to be an open subset of a properly convex cone in an affine subspace A^n and the end fundamental group acts on it as a discrete linear group of determinant 1. Then the theory of convex cones applies and using the level sets of the Koszul-Vinberg function we obtain a smooth convex one-sided neighborhood in U (see Lemma 6.5 and 6.6 of Goldman [30]). Also, the outer one-sided neighborhood can be obtained by a reflection about the plane containing $\tilde{\Sigma}$ and the end vertex. $\square$

Proposition 2.2. *Let $\mathcal{O}$ be a convex real projective 3-orbifold with the end orbifolds each of which is homeomorphic to a sphere $S_{3,3,3}$ or a disk with three silvered edges and three vertices of edge orders $3, 3, 3$. Then the orbifold has lens-shaped or horospherical ends.*

Proof. Let E be an end of type $S_{3,3,3}$ for $\tilde{\mathcal{O}}$. It is sufficient to consider only $S_{3,3,3}$ since it double-covers the disk orbifold. Since $\mathbf{\Gamma}_E$ is generated by finite order elements fixing an end vertex $\mathbf{v}_E$, it follows that every holonomy element has eigenvalue equal to 1 at $\mathbf{v}_E$. Take a finite-index free abelian group A of rank two. Hence a convex projective torus T^2 covers E. Therefore, Ω_E projectively diffeomorphic either to a complete affine space or to the interior of a properly convex triangle or to a half-space. We can easily show that the end orbifold admits a complete affine structure or is a quotient of a properly convex triangle as it cannot be a quotient of a half-space with a distinguished foliation by lines by the existence of holonomy of order 3 fixing a point.

If Σ_E has a complete affine structure, we have a horospherical end for E by Theorem 3.9. Suppose that Σ_E has a triangle as its universal cover. A acts with an element g' with an eigenvalue > 1 and an eigenvalue < 1 as a transformation in $\mathsf{SL}_{\pm}(3, \mathbb{R})$ the group of projective automorphisms at $\mathbb{S}^2_{\mathbf{v}}$. This means that there are fixed points v_1 and v_2 of g' other than $\mathbf{v}_E$ in a direction of the vertices of the triangle in the cone. Since the corresponding eigenvalue at $\mathbf{v}_E$ is 1 and g' acts on a properly convex domain, we see that g' has four fixed points and an invariant subspace P disjoint from $\mathbf{v}_E$. Since elements of A commute with g', so does every other $g \in A$. It follows that the end

fundamental group acts on P as well. We have a totally geodesic conical end and by Theorem 3.13, the end is lens-shaped. $\square$

Example 2.3 (Lee's example). Consider the ideal hyperbolic structure on a cube P with all sides mirrored and all edges given order 3 but vertices removed. By the Mostow-Prasad rigidity and the Andreev theorem, there exists a unique hyperbolic structure on it. There exists a six-dimensional space of real projective structures on it as found in [21] where one has projectively fixed fundamental domain in the universal cover $\tilde{P}$.

There are eight vertices of P corresponding to eight ends of P. Each end is a 2-orbifold based on a triangle with edges mirrored and vertex orders are all 3. Thus, each end has a neighborhood homeomorphic to the 2-orbifold multiplied by $(0, 1)$. We can characterize them by a real-valued invariant. Their invariants are related when we are working on the restricted deformation space. (They might be independent as M. Davis and R. Green observed in the full deformation space.)

Then this end can be a horospherical or is of lens type with a totally geodesic realization end orbifold in it by Proposition 2.2. When P is hyperbolic, the ends are horospherical as P has a complete hyperbolic structure.

Experimentations suggest that we realize totally geodesic conical ends after deformations. This applies to S. Tillman's example. (See my other paper on the mathematics archive [13] for details)

We also discussed bending:

Example 2.4. Let $\mathcal{O}$ have the usual assumptions. Let the associated orbifold Σ_E for an end E of $\mathcal{O}$ be a closed 2-orbifold of negative orbifold Euler characteristic. Let c be a simple closed geodesic. E has an end neighborhood U in $\mathcal{O}$ diffeomorphic to $\Sigma_E \times (0, 1)$. c has an embedded annulus A diffeomorphic to $c \times (0, 1)$ foliated by radial lines. Suppose that the end E is totally geodesic conical and of lens type with all eigenvalues 1 at the end vertex. Let U be an end neighborhood in $\tilde{\mathcal{O}}$ corresponding to E and $\mathbf{v}_E$ denotes the corresponding end vertex. Let g_c be the deck transformation corresponding to c and it acts on a geodesic l_c in $R_{\mathbf{v}_E}(E)$. Let A' denote the disk in U corresponding to l_c. Then the holonomy g_c is conjugate to a diagonal matrix with entries $\lambda, \lambda^{-1}, 1, 1$ where the last 1 corresponds to the vertex $\mathbf{v}$. We take an element k_b of $\mathsf{SL}_{\pm}(4, \mathbb{R})$ of form in this system of coordinates

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & b & 1 \end{pmatrix} \tag{2}$$

where $b \in \mathbb{R}$. k_b commutes with g_c and we can "bend" E by k_b: Define $\tilde{A} = \bigcup_{\gamma \in \mathbf{\Gamma}_E} \gamma(A')$. Then we cut U by $\tilde{A}$ and we obtain two copies A_1 and A_2 corresponding to a single component by completing $U - \tilde{A}$. We take an ambient real projective manifold U' containing the completion. We can find neighborhoods N_1 and N_2 of A_1 and A_2 in U diffeomorphic by a projective map $\hat{k}_b$ induced by k_b.

We take a disjoint union $(U - \tilde{A}) \coprod N_1 \coprod N_2$ and quotient it by identifying elements of N_1 with elements near A_1 in $U - \tilde{A}$ by the identity map and elements of N_2 with

elements near A_2 in $U - \tilde{A}$ by the identity also. We then glue back N_1 and N_2 by $\hat{k}_b$ the real projective diffeomorphism of a neighborhood of N_1 to that of N_2. We will do this for every such pair using appropriate conjugates of $\hat{k}_b$. This construction is called "bending" and was investigated by Johnson and Millson [36].

For sufficiently small b, we see that the end is still of lens type and it is not totally geodesic conical. (This follows since the condition of being of lens type is an open condition. See Theorem 3.15.)

Since k_b fixes a subspace of dimension 2 containing $\mathbf{v}_E$ and the geodesic fixed by g_c, the totally geodesic subspace is bent. We see that $b > 0$, we obtain a boundary of E bent in a positive manner. The deformed holonomy group acts on a convex domain obtained by bendings of these types everywhere.

For the same c, let k_s be given by

$$\begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1/s^3 \end{pmatrix} \tag{3}$$

where $s \in \mathbb{R}^+$. These give us bendings of second type. (We talked about this in [13].) For s sufficiently close to 1, the properly of being lens is preserved and being totally geodesic conical. However, these will be understood by homology so that we will not be needing this.

If $s\lambda < 1$ for the maximal eigenvalue λ of a closed curve meeting c odd number of times, we have that c has the attacking fixed point at $\mathbf{v}_E$. This implies that we no longer have lens-type ends if we have started with a lens-shaped end. .

3. End theory

In this section, we discuss the properties of horospherical and lens-shaped ends. These results will be need in later papers [13].

3.1. **The holonomy homomorhisms of the end fundamental groups: the fibering.** Let E be an end of $\tilde{\mathcal{O}}$. Let $\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbf{v}_E}$ be the subgroup of $\mathsf{SL}_\pm(n+1,\mathbb{R})$ fixing a point $\mathbf{v}_E \in \mathbb{S}^n$. This group can be understood as follows by letting $\mathbf{v}_E = [0,\dots,0,1]$ as a group of matrices: For $g \in \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbf{v}_E}$, we have

$$\begin{pmatrix} \frac{1}{\lambda_{\mathbf{v}_E}(g)^{1/n}}\hat{h}(g) & \vec{0} \\ \vec{v}_g & \lambda_{\mathbf{v}_E}(g) \end{pmatrix}$$

where $\hat{h}(g) \in \mathsf{SL}_\pm(n,\mathbb{R})$, $\vec{v} \in \mathbb{R}^{n*}$, $\lambda_{\mathbf{v}_E}(g) \in \mathbb{R}_+$, so-called the linear part of h. Here,

$$\lambda_{\mathbf{v}_E} : g \mapsto \lambda_{\mathbf{v}_E}(g) \text{ for } g \in \mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbf{v}_E}$$

is a homomorphism so it is trivial in the commutator group $[\Gamma_E,\Gamma_E]$. There is a group homomorphism $\mathcal{L}' : \mathsf{SL}_\pm(n+1,\mathbb{R}) \to \mathsf{SL}_\pm(n,\mathbb{R}) \times \mathbb{R}_+$ by sending the above matrix to $(\hat{h},\lambda)$ with kernel $\mathbb{R}^{n*}$ a dual space to $\mathbb{R}^n$. Thus, we obtain a diffeomorphism

$$\mathsf{SL}_\pm(n+1,\mathbb{R})_{\mathbf{v}_E} \to \mathsf{SL}_\pm(n,\mathbb{R}) \times \mathbb{R}^{n*} \times \mathbb{R}_+.$$

(We denote by $\mathcal{L}$ the further projection to $\mathsf{SL}_\pm(n,\mathbb{R})$.)

Let Σ_E be the end $(n-1)$-orbifold. Given a representation $\hat{h} : \pi_1(\Sigma_E) \to \mathsf{SL}_\pm(n, \mathbb{R})$ and $\lambda : \pi_1(\Sigma_E) \to \mathbb{R}^+$, we denote by $\mathbb{R}^n_{\hat{h},\lambda}$ the $\mathbb{R}$-module with $\pi_1(\Sigma_E)$-action given by $g \cdot \vec{v} = \frac{1}{\lambda(g)^{1/n}} \hat{h}(g)(\vec{v})$. And $\mathbb{R}^{n*}_{\hat{h},\lambda}$ will be the dual vector space.

The space of representations

$$\mathsf{Hom}(\pi_1(\Sigma_E), \mathsf{SL}_\pm(n+1, \mathbb{R})_{\mathbf{v}_E})/\mathsf{SL}_\pm(n+1, \mathbb{R})_{\mathbf{v}_E}$$

can be understood by mapping into

$$\mathsf{Hom}(\pi_1(\Sigma_E), \mathsf{SL}_\pm(n+1, \mathbb{R}))/\mathsf{SL}_\pm(n, \mathbb{R}) \times \mathsf{Hom}(\pi_1(\Sigma_E), \mathbb{R}_+)$$

where the fiber over $(\hat{h}, \lambda)$ is $H^1(\pi_1\Sigma_E, \mathbb{R}^{n*}_{\hat{h},\lambda})$ for

$$(\hat{h}, \lambda) \in \mathsf{Hom}(\pi_1(\Sigma_E), \mathsf{SL}_\pm(n+1, \mathbb{R}))/\mathsf{SL}_\pm(n, \mathbb{R}) \times \mathsf{Hom}(\pi_1(\Sigma_E), \mathbb{R}_+)$$

where $\vec{v}$ is determined by elements of $H^1(\pi_1(E), \mathbb{R}^{n*}_{\hat{h},\lambda})$ where $\mathbb{R}^{n*}_{\hat{h},\lambda}$ is a dual space of $\mathbb{R}^n_{\hat{h},\lambda}$; i.e., as a cocyle:

$$(A, \vec{v}, \lambda)(B, \vec{w}, \mu) = (AB, \frac{1}{\mu^{1/n}} \vec{v}B + \lambda \vec{w}, \lambda\mu).$$

Note $\mathsf{Hom}(\pi_1(\Sigma_E), \mathbb{R}_+)$ equals $H^1(\pi_1(\Sigma_E), \mathbb{R})$.

Theorem 3.1. *Let $\mathcal{O}$ a properly convex real projective orbifold with radial ends and let $\tilde{\mathcal{O}}$ be its universal cover. Let Σ_E be the end orbifold associated with an end E of $\tilde{\mathcal{O}}$. The space of representations of the end fundamental group $\pi_1(\Sigma_E)$ to*

$$\mathsf{SL}_\pm(n+1, \mathbb{R})_{\mathbf{v}_E}$$

is the fiber space B over

$$\mathsf{Hom}(\pi_1(\Sigma_E), \mathsf{SL}_\pm(n, \mathbb{R}))/\mathsf{SL}_\pm(n, \mathbb{R}) \times H^1(\pi_1(\Sigma_E), \mathbb{R}_+)$$

with the fiber equal to $H^1(\pi_1(\Sigma_E), \mathbb{R}^{n*}_{\hat{h},\lambda})$ for each

$$(\hat{h}, \lambda) \in \mathsf{Hom}(\pi_1(\Sigma_E), \mathsf{SL}_\pm(n, \mathbb{R}))/\mathsf{SL}_\pm(n, \mathbb{R}) \times H^1(\pi_1(\Sigma_E), \mathbb{R}).$$

We remark that we don't really understand the fiber dimensions and their behavior as we change the base points. A similar idea is given by Mess [46]. In fact the dualizing these matrices gives us a representation to $\mathbf{Aff}(A^n)$. In particular if we restrict ourselves to linear parts to be in $\mathsf{SO}(n, 1)$, then we are exactly in the cases studied by Mess. (See the concept of the duality in Subsection 3.3 and Appendix A.)

3.2. **Tubular actions.** Let us give a pair of antipodal points $\mathbf{v}$ and $\mathbf{v}_-$. If a group Γ of projective automorphisms fixes a pair of fixed points $\mathbf{v}$ and $\mathbf{v}_-$, then Γ is said to be *tubular*. There is a projection $\Pi_{\mathbf{v}} : \mathbb{S}^n - \{\mathbf{v}, \mathbf{v}_-\} \to \mathbb{S}^{n-1}_{\mathbf{v}}$ given by sending every ray with endpoints $\mathbf{v}$ and $\mathbf{v}_-$ to the sphere of directions at $\mathbf{v}$. A convex tube in $\mathbb{S}^n$ is the closure of the inverse image of a convex domain Ω in $\mathbb{S}^{n-1}_{\mathbf{v}}$. Given an end E of $\tilde{\mathcal{O}}$, the *end domain* is $R_{\mathbf{v}}(\tilde{\mathcal{O}})$. If an end E has the end domain Ω_E on which the end fundamental group $\pi_1(E)$ is acting, it follows that $h(\pi_1(E))$ acts on the tube domain $\mathcal{T}_E$ associated with Ω_E.

Letting $\mathbf{v}$ have the coordinates $[0, \dots, 0, 1]$, the matrix of g of $\pi_1(E)$ is of form

$$\begin{pmatrix} \frac{1}{\lambda_{\mathbf{v}}(g)^{\frac{1}{n}}} \hat{h}(g) & 0 \\ \vec{b}_g & \lambda_{\mathbf{v}}(g) \end{pmatrix} \tag{4}$$

where $\vec{b}_g$ is an $n \times 1$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant ± 1 and $\lambda_{\mathbf{v}}(g)$ is a positive constant.

Note that the representation $\hat{h} : \pi_1(E) \to \mathsf{SL}_{\pm}(n, \mathbb{R})$ is given by sending $g \mapsto \hat{h}(g)$. Here we have $\lambda_{\mathbf{v}}(g) > 0$ and $\det \hat{h}(g) > 0$ always if Σ_E is orientable. If Ω_E is properly convex, then we say that the convex tubular domain is *properly tubular* and the action is said to be *properly tubular*.

3.3. **Affine actions dual to tubular actions.** Let us consider the dual projective space $\mathcal{P}(\mathbb{R}^{n+1*})$. We can identify it with $\mathcal{P}(\mathbb{R}^{n+1})$ using a nondegenerate symmetric bilinear form $B : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}$. We can fix one here. This gives an isomorphism $D^*_B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1*}$ by sending $v \mapsto B(v, \cdot) \in \mathbb{R}^{n+1*}$.

A linear transformation $L : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ induces one $L^* : \mathbb{R}^{n+1*} \to \mathbb{R}^{n+1*}$ by sending $\alpha \mapsto \alpha \circ L$ for an element α of $\mathbb{R}^{n+1*}$. We define the dual linear map $L^{*'} = D^{*,-1}_B \circ L^* \circ D^*_B : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$. Thus, every projective automorphism $g : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1})$ has a dual $g^* : \mathcal{S}(\mathbb{R}^{n+1,*}) \to \mathcal{S}(\mathbb{R}^{n+1,*})$ and the dual $g^{*,'} : \mathcal{S}(\mathbb{R}^{n+1}) \to \mathcal{S}(\mathbb{R}^{n+1})$.

We will identify $\mathbb{R}^{n+1}$ and $\mathbb{R}^{n+1*}$ by a fixed D^*_B and correspondingly for $\mathcal{S}(\mathbb{R}^{n+1})$ and $\mathcal{S}(\mathbb{R}^{n+1*})$. Then $g^{*'} = g^*$. We also note that $(g^*)^* = g$.

The automorphism preserving a codimension-one subspace $\mathbb{S}^{n-1}_\infty$ of $\mathcal{S}(\mathbb{R}^{n+1})$ and the components of the complement acts on an affine space A^n that is a component of the complement of $\mathbb{S}^{n-1}_\infty$. The subgroup of projective automorphisms preserving $\mathbb{S}^{n-1}_\infty$ and the components equals $\mathbf{Aff}(A^n)$.

By duality, a great $(n-1)$-sphere $\mathbb{S}^{n-1}_\infty$ corresponds to a point $\mathbf{v}_{\mathbb{S}^{n-1}_\infty}$. Thus, given a group Γ in $\mathbf{Aff}(A^n)$, we obtain dual groups Γ^* acting on $\mathcal{S}(\mathbb{R}^{n+1,*})$ and $\Gamma^{*,'}$ acting on $\mathcal{S}(\mathbb{R}^{n+1})$. Here, of course $\Gamma^* = \Gamma^{*,'}$ and $(\Gamma^*)^* = \Gamma$.

An open convex cone C in $\mathbb{R}^{n+1}$ is *dual* to an open convex cone C^* in $\mathbb{R}^{n+1,*}$ if $C^* \subset \mathbb{R}^{n+1*}$ is the set of linear transformations taking positive values on $\mathrm{Cl}(C)$. C^* is a cone with vertex as the origin again. Note $(C^*)^* = C$. A properly convex open domain Ω in $\mathcal{S}(\mathbb{R}^{n+1})$ is *dual* to a properly convex open domain Ω^* in $\mathcal{S}(\mathbb{R}^{n+1,*})$ if Ω corresponds to an open convex cone C and Ω^* to its dual C^*. We say that Ω^* is dual to Ω. We also have $(\Omega^*)^* = \Omega$ and Ω is properly convex if and only if so is Ω^*.

For a closed cone C, the closure of the set $f \in \mathbb{R}^{n+1,*}$ taking nonnegative values on C is defined as the dual $C^* \subset \mathbb{R}^{n*}$ of C. For compact convex subset Ω in $\mathbb{S}^{n-1}$, the dual domain is defined as the quotient of the dual cone of the cone corresponding to Ω. The dual set is also compact and convex but the dimension may not be change. Again, we have $(\Omega^*)^* = \Omega$.

The linear part $\mathcal{L}(\Gamma)$ acts on a properly convex cone C in $\mathbb{R}^{n+1}$ if and only if its dual linear group $\mathcal{L}(\Gamma)^*$ acts on a properly convex domain C^* dual to C. Note that Γ acts on Ω if and only if the dual group Γ^* acts on Ω^*.

Suppose that Γ acts on a properly convex open domain Ω in $\mathbb{S}^{n-1}_\infty$. Then we call Γ an asymptotically *properly convex affine* action. Then the dual group Γ^* acts on

a properly tubular domain with vertices $\mathbf{v}_{\mathbb{S}^{n-1}_\infty}$ and $\mathbf{v}_{\mathbb{S}^{n-1}_\infty,-}$. We can sum up that the tubular domain $\Omega \subset \mathbb{S}^{n-1}_\infty$ corresponds to the dual domain $\Omega^* \subset \mathbb{S}^n_{\mathbf{v}_{\mathbb{S}^{n-1}_\infty}}$. Finally, there is a group isomorphism

$$\Gamma \to \Gamma^* \text{ given by } g \mapsto g^{*,-1}.$$

Definition 3.2. Let $\mathbf{d}$ denote the standard spherical metric on $\mathbb{S}^n$. Given two compact subsets K_1 and K_2 of $\mathbb{S}^n$, we define the spherical distance $\mathbf{d}_H(K_1, K_2)$ between K_1 and K_2 to be

$$\inf\{\epsilon > 0 | K_2 \subset N_\epsilon(K_1), K_1 \subset N_\epsilon(K_2)\}.$$

The simple distance $\mathbf{d}(K_1, K_2)$ is defined as

$$\inf\{\mathbf{d}(x, y) | x \in K_1, K_2\}.$$

Definition 3.3. A properly tubular action is said to be *distanced* if there exists a properly convex compact Γ-invariant subset in the tubular domain disjoint from the vertices. A properly convex affine action of Γ is said to be *asymptotically nice* if there exists a properly convex Γ-invariant open domain U' in A^n with boundary in $\Omega \subset \mathbb{S}^{n-1}_\infty$ and U' is in the intersection of all open hemispheres H supporting U' at $\mathrm{bd}\Omega$ and A^n where for each point $x \in \mathrm{bd}\Omega$ there exists a supporting hemisphere H_x with $\mathrm{bd} H_x$ distinct from $\mathbb{S}^{n-1}_\infty$ and containing x. An *asymptotic* hypersurface is the boundary of the minimal half-space supporting U at a point of $\mathrm{bd}\Omega$.

Proposition 3.4. *Let Γ and Γ^* be dual groups where Γ has an affine action on A^n and Γ^* is tubular with the vertex $\mathbf{v} = \mathbf{v}_{\mathbb{S}^{n-1}_\infty}$ dual to the boundary $\mathbb{S}^{n-1}_\infty$ of A^n $\Gamma = (\Gamma^*)^*$ acts on a properly convex domain V in A^n and is asymptotically nice if and only if Γ^* acts on a properly tubular domain B and is distanced. Moreover, the minimal Γ^*-invariant compact subset in B is uniquely determined if and only if Γ has unique asymptotic hypersurface at each point of the boundary of Ω.*

Proof. For each point x of $\mathrm{bd}\Omega$, there exists an open hemisphere in $\mathbb{S}^n$ at x supporting V uniformly bounded at a distance in the $\mathbf{d}_H$-sense from the open hemisphere A^n with boundary $\mathbb{S}^{n-1}_\infty$. (Otherwise, V would become empty.) For each $x \in \mathrm{bd}\Omega$, we choose the supporting hemisphere $H_\mathbf{v}$ so that $H_\mathbf{v} \cap A^n$ is smallest.

The dual points of the supporting hyperplane are points on ∂B for the dual tube domain B. Since the hyperspheres of form $H_\mathbf{v}$, $x \in \mathrm{bd}\Omega$, are bounded at a distance from $\mathbb{S}^{n-1}_\infty$ in the $\mathbf{d}_H$-sense, the dual points are uniformly bounded at a distance from the vertices $\mathbf{v}$ and $\mathbf{v}_-$. Let us call this set K. Then for every point of $\mathrm{bd}\Omega^*$ the boundary of the dual $\Omega^* \subset \mathbb{S}^{n-1}_\mathbf{v}$ of Ω, we have a point of K in the corresponding ray from $\mathbf{v}$ to $\mathbf{v}_-$. K is uniformly bounded at a distance from $\mathbf{v}$ and $\mathbf{v}_-$ in the $\mathbf{d}$-sense. The convex hull of K is a compact convex set bounded at a distance from $\mathbf{v}$ and $\mathbf{v}_-$ since the tube domain is properly convex. Since K is Γ-invariant, so is the convex hull. Conversely, every compact convex subset K of the tubular domain B bounded away from $\mathbf{v}$ and $\mathbf{v}_-$ meet a ray from $\mathbf{v}$ to $\mathbf{v}_-$ at a point bounded away from the end points. Let A' denote the set $\partial B - \{\mathbf{v}, \mathbf{v}_-\}$. Then $K \cap A'$ is a compact convex and Γ-invariant and bounded away from $\mathbf{v}, \mathbf{v}_-$. We denote by K' the boundary of a component $A' - K$ containing $\mathbf{v}$ in its closure. Then K' is again Γ-invariant and each ray from $\mathbf{v}$ to $\mathbf{v}_-$ meets K' at a unique point.

Each point x of K' is dual to a hypersphere P in $\mathbb{S}^n$ bounded at a distance from $\mathbb{S}^{n-1}_\infty$ since x is bounded at a distance from $\mathbf{v}, \mathbf{v}_-$. Since $x \in \partial B$, P must be a supporting plane to the dual of B, a convex domain Ω in $\mathbb{S}^{n-1}_\infty$; and $P \cap A^n$ is a complete hyperplane with a point of $\mathrm{bd}\Omega$ its boundary in $\mathbb{S}^n$. The intersection of the corresponding half-spaces in A^n is a convex open domain.

Suppose that Γ^*-invariant compact subset K in B is uniquely determined. Then K meets each complete segment in ∂B at a unique point. If not, we can form a convex hull from the so-called lower boundary of K given as the boundary component of $\partial B - K$ containing $\mathbf{v}$ or from the so-called upper boundary of K given as that of $\partial B - K$ containing $\mathbf{v}_-$.

The unique point of $K \cap s$ for a complete segment s in ∂B corresponding to $q \in \mathrm{bd}\Omega$ gives us the unique asymptotic hyperplane to q.

The converse is also very simple to prove. □

Theorem 3.5. *Let Γ be a nontrivial properly convex tubular action at vertex $\mathbf{v} = \mathbf{v}_{\mathbb{S}^{n-1}_\infty}$ and acts on a properly convex domain U and satisfy the weakly uniform middle-eigenvalue conditions. Then Γ is distanced inside the tube B where Γ acts on. Furthermore, if Γ satisfies the uniform middle-eigenvalue conditions, then the distanced Γ-invariant set K in B is uniquely determined.*

Proof. Let $\mathbf{v}$ be the vertex of the tube B. First assume that Γ induces an irreducible action on the link sphere $\mathbb{S}^{n-1}_{\mathbf{v}}$. The dual group Γ^* acts on a properly convex domain $U^* \subset \mathbb{S}^n$ dual to U. Then the closure of U^* meets $\mathbb{S}^{n-1}_\infty$ in a domain Ω^* dual to the convex domain in $\mathbb{S}^{n-1}_{\mathbf{v}}$ corresponding to the tube of Γ. By Theorem A.1, Γ^* is asymptotically nice. Proposition 3.4 implies the result.

Suppose that Γ acts reducibly on $\mathbb{S}^{n-1}_{\mathbf{v}}$. Then Γ is isomorphic to $\mathbb{Z}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$ where Γ_i is nontrivial hyperbolic for $i = 1, \dots, s$ and trivial for $s+1 \leq i \leq l_0$ where $s \leq l_0$. Each Γ_i for $i = 1, \dots, s$ acts on a nontrivial tube B_i with vertices $\mathbf{v}$ and $\mathbf{v}_-$. By above it is a distanced action and it acts on a Γ_i-invariant compact convex set $K_i \subset B_i$ disjoint from $\{\mathbf{v}, \mathbf{v}_-\}$.

For each i, $s+1 \leq i \leq r$, there exists a complete segment B_i with endpoints $\mathbf{v}$ and $\mathbf{v}_-$. A point p_i correspond to B_i in $\mathbb{S}^{n-1}_{\mathbf{v}}$. For g in the center, we have $\lambda_1(g) > \lambda_{\mathbf{v}}(g)$ or $\lambda_1(g) = \lambda_{\mathbf{v}}$ for the eigenvalues in $\mathbb{S}^n$. Hence, there exists a fixed point p_i' for some central g in B_i where $\lambda_1(g) > \lambda_{\mathbf{v}}(g)$ holds or $\lambda_1(g) = \lambda_{\mathbf{v}}(g)$ for every central g. In the later case we choose p_i' to be an arbitrary point of the interior of B_i. By commutativity, p_i' is a fixed point of every element of $\mathbb{Z}^{s-1}$.

The convex hull of
$$K_1 \cup \cdots \cup K_s \cup \{p'_{s+1}, \dots, p'_{l_0}\}$$
is a distanced Γ-invariant compact convex set.

Suppose that Γ satisfies the uniform middle-eigenvalue conditions, Then K_i for $i = 1, \dots, s$ is unique by Theorem A.1 and Proposition 3.4. Also, p_i' is uniquely determined for $i \geq s+1$. □

3.4. **Duality to totally geodesic ends and properly convex neighborhood.** The two subgroups G_1 of Γ and G_2 of Γ^* are *dual* if sending $g \to g^{-1,T}$ gives us a one-to-one map $G_1 \to G_2$. A *totally geodesic end* of a real projective orbifold is given

as an end that has an end neighborhood that can be compactified to be an orbifold with totally geodesic boundary component corresponding to the end. An *end orbifold* of a totally geodesic end is the boundary component. (This is a distinct notion from totally geodesic conical end.)

There exists a one-to-one correspondence between ends of $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}^*$ by considering the dual relationship Γ_E and $\Gamma^*_{E'}$ for ends E and E'. This follows since there are exiting sequence of submanifolds $\Sigma_{E,i}$ for E where Γ_E acts cocompactly and $\Sigma_{E',j}$ where $\Gamma^*_{E'}$ acts cocompactly. (We omit here the topological detailed proof.)

Also, it is elementary to check that a horospherical end is dual to a horospherical end. Thus, the horospherical nature of ends is a self-dual notion.

Proposition 3.6. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. Then the dual real projective orbifold $\mathcal{O}^*$ has the same number of ends so that*

- *there exist a one-to-one correspondence $\mathcal{C}$ between the set of ends of $\mathcal{O}$ and the set of ends of $\mathcal{O}^*$.*
- *$\mathcal{C}$ restrict to such a one between the subet of horospherical ends of $\mathcal{O}$ and the subset of horospherical ones of $\mathcal{O}^*$.*
- *$\mathcal{C}$ restrict to such a one between the subset of properly convex radial ends of $\mathcal{O}$ and the subset of totally geodesic ones of $\mathcal{O}^*$.*
- *$\mathcal{C}$ restrict to such a one between the subset of totally geodesic ends of $\mathcal{O}$ with the subset of ends of radial ones of $\mathcal{O}^*$.*

Proof. Let $\tilde{\mathcal{O}}$ be the universal cover of $\mathcal{O}$. Let $\tilde{\mathcal{O}}^*$ be the dual domain. It is sufficient to prove the above correspondence for ends of $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}^*$. Let $C \subset \mathbb{R}^{n+1}$ and $C^* \subset \mathbb{R}^{n+1*}$ denote the respective cones. Each end vertex x of $\tilde{\mathcal{O}}$ has a supporting hyperplanes of C whose supporting linear functions form a properly convex domain of dimension $n-1$ if x corresponds to a properly convex end. The supporting hyperspace is unique if x is horospherical or totally geodesic. The intersection of the supporting hyperspace with the closure of $\tilde{\mathcal{O}}$ is unique if and only if x is horospherical. □

See Section A for the definition of uniform middle-eigenvalue condition for affine actions.

Proposition 3.7. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. The following conditions are equivalent:*

(i) *A properly convex radial end of $\mathcal{O}$ satisfies the uniform middle-eigenvalue condition.*
(ii) *The corresponding totally geodesic end of $\mathcal{O}^*$ satisfies this condition.*
(iii) *The end surface of the corresponding totally geodesic end of $\mathcal{O}^*$ has a properly and strictly convex neighborhood in some ambient real projective orbifold $\mathcal{O}'$ containing $\mathcal{O}^*$.*

Proof. The items (i) and (ii) are equivalent by considering equation 4.

We prove the equivalence of items (i) and (iii). Let E denote a radial end of $\tilde{\mathcal{O}}$ and E^* denote the corresponding end of the dual domain $\tilde{\mathcal{O}}^*$ of $\tilde{\mathcal{O}}$. Let Γ_E denote the subgroup of Γ corresponding to E and Γ_{E^*} the subgroup of Γ^* corresponding to E^*.

The condition (i) implies that there exists a distanced Γ_E-invariant compact set by Theorem 3.5. Proposition 3.4 shows that there exists an asymptotically nice properly convex set U in the affine space A^n. Also, there exists a properly convex open set Ω_{E^*} in $\mathrm{bd} A^n$ invariant under Γ_{E^*}. In the proof of Lemma A.9, we form $H(x)$ for $x \in \mathrm{bd}\Omega_{E^*}$. It is easy to see that $\bigcap_{x \in \mathrm{bd}\Omega_{E^*}} H(x)$ is a properly convex domain $\hat{U}$ and $\hat{U}/\mathbf{\Gamma}_{E^*}$ is a properly convex open neighborhood of $\Omega_{E^*}/\mathbf{\Gamma}_{E^*}$. The proof showing that this contains a strictly convex open neighborhood is entirely similar to one in the proof (ii) $\Rightarrow$ (i) of Theorem 4.5. By attaching this neighborhood to $\mathcal{O}$, we obtain the third item.

Assuming (iii), there exists a totally geodesic $(n-1)$-dimensional properly convex domain Ω_{E^*} in a subspace $\mathbb{S}^{n-1}$ where $\mathbf{\Gamma}_{E^*}$ acts on. Let U be the two-sided properly convex neighborhood of it where $\mathbf{\Gamma}_{E^*}$ acts on. Then it follows that the supporting hemisphere at each point of $\mathrm{bd}\Omega_{E^*}$ is now transversal to $\mathbb{S}^{n-1}$. Now the proof is similar to the proof (i) $\Rightarrow$ (ii) of Theorem 4.5. $\square$

We remark that the notion of totally geodesic conical end is "self-dual" in the sense that the dual of a totally geodesic conical end is a totally geodesic end that can be extended to be made into a totally geodesic conical end by taking cone over a fixed point. This follows since such an end has to have an invariant totally geodesic subspace of codimension one and a fixed point outside it.

3.5. **Horospherical ends.** In this section, we will try to compile the important properties of horospherical ends needed here.

By an *exiting sequence* of end neighborhoods U_i of $\tilde{\mathcal{O}}$, we mean a sequence of end neighborhoods U_i so that $p(U_i) \subset \mathcal{O}$ is so that for each compact subset K of $\mathcal{O}$, $p(U_i) \cap K \neq \emptyset$ for only finitely many is.

Proposition 3.8. *Let $\mathcal{O}$ be a properly convex real projective n-orbifold with radial ends. Let E be a horospherical end of its universal cover $\tilde{\mathcal{O}}$ and $\mathbf{\Gamma}_E$ denote the end fundamental group.*

(i) *The space $\Omega_E := R_{\mathbf{v}_E}(\tilde{\mathcal{O}})$ of rays from the end point $\mathbf{v}_E$ forms a complete affine space of dimension $n-1$.*
(ii) *The norms of eigenvalues of $g \in \mathbf{\Gamma}_E$ are all 1.*
(iii) *An end point of a horospherical end cannot be on a segment in $\mathrm{bd}\tilde{\mathcal{O}}$.*
(iv) *For any compact set K' inside a horospherical neighborhood, there exists a smooth convex smooth neighborhood disjoint from K'.*
(v) *$\pi_1(E)$ is virtually nilpotent.*

Proof. Let U be a horospherical end with an end vertex $\mathbf{v}_E$. The space of rays from the end vertex forms a convex subset Ω_E of a complete affine space $\mathbb{R}^{n-1} \subset \mathbb{S}^{n-1}$ and covers an end orbifold Σ_E with the discrete group $\pi_1(E)$ acting as a discrete subgroup Γ'_E of the projective automorphisms so that Ω_E/Γ'_E is projectively isomorphic to Σ_E.

We prove the items (i) and (ii). From the paper [10], we see that Ω_E is properly convex or is foliated by complete affine spaces of dimension i_0 with the common boundary sphere of dimension $i_0 - 1$ and the space of the leaves forms a properly open convex subset K of $\mathbb{S}^{n-i_0-1}$. (See Section 1.4 of [10].) Then $\mathbf{\Gamma}_E$ acts on K cocompactly but perhaps not discretely. Here, we aim to show $i_0 = n-1$ and K is a point.

Suppose that $i_0 \leq n-2$. This implies that Ω_E is foliated by complete affine spaces of dimension $i_0 \leq n-2$.

For each element g of $\mathbf{\Gamma}_E$, a complex or negative eigenvalue of g in $\mathbb{C}-\mathbb{R}^+$ cannot have a maximal or minimal absolute value different from 1 since otherwise by taking the convex hull of $\{g^m(x)|m \in \mathbb{Z}\}$ for a point x of U, we see that U must be not properly convex. Thus, the largest and the smallest absolute value eigenvalues of g are positive.

Since $\mathbf{\Gamma}_E$ acts on a properly convex subset K of dimension ≥ 1, it follows that an element g has an eigenvalue > 1 and an eigenvalue < 1 by Benoist [1] as an element of projective automorphism on the minimal great sphere containing K. Hence, there exists the largest norm of eigenvalues and the smallest one of g in $\mathbf{Aut}(\mathbb{S}^n)$ both different from 1. Therefore, let $\lambda_1 > 1$ be the greatest norm eigenvalue and $\lambda_2 < 1$ be the smallest norm one of this element g. Let $\lambda_0 > 0$ be the eigenvalue of g associated with $\mathbf{v}_E$. These are all positive. The possibilities are as follows

$$\lambda_1 = \lambda_0 > \lambda_2, \lambda_1 > \lambda_0 > \lambda_2, \lambda_1 > \lambda_2 = \lambda_0.$$

In all cases, at least one of the largest norm or the smallest norm is different from λ_0. Thus there exists a corresponding fixed point x_∞ distinct from $\mathbf{v}_E$ with the distinct eigenvalue from λ_0. We have $x_\infty \in \mathrm{Cl}(U)$ since x_∞ is a limit of $g^i(x)$ for $i \to \infty$ or $i \to -\infty$. Since $x_\infty \notin U$, it follows that $x_\infty = \mathbf{v}_E$ by the definition of the horoballs. This is a contradiction.

From here it follows that $\lambda_j = 1$ for each norm λ_j of eigenvalues of g for every $g \in \mathbf{\Gamma}_E$. Hence, K is a point since otherwise there has to be some eigenvalue with norm > 1. This implies that we have $i = n-1$ and Ω_E is a complete affine space by our theory in [10]. The second item is clear now.

(iii) Suppose that $\mathbf{v}_E$ is an endpoint for a segment s in $\mathrm{bd}\tilde{\mathcal{O}}$. Let us choose a maximal two-dimensional disk D containing s with $D^o \subset \tilde{\mathcal{O}}^o$. Then $D \cap \partial U$ contains an arc α so that α converges to $\mathbf{v}_E$ in one direction. Let $p_i \in \alpha$ be a point converging to $\mathbf{v}_E$. There exists a deck transformation g_i in the end fundamental group at $\mathbf{v}_E$ so that $g_i(p_i)$ is in a fixed compact fundamental domain F of ∂U. Then g_i fixes the end vertex $\mathbf{v}_E$ with eigenvalue 1 and g_i decomposes $\mathbb{R}^{n+1}$ into blocks of matrix with diagonal entries 1 and the entries above it all 1 and other entries zero and blocks of identity matrix and blocks of complex eigenvalues of absolute values 1. There is a subspace P where g_i restrict to the union of blocks of identity and complex unit eigenvalues.volume Let $\{r_i\}$ be a sequence of bounded projective automorphisms fixing $\mathbf{v}_E$ and $r_i \circ g_i(D)$ is in the plane containing D and $r_i \circ g_i(s)$ is in the line containing s. (We do not require r_i to be in $\mathbf{\Gamma}$.) Notice that α is differentiable at $\mathbf{v}_E$ since it is strictly convex. Since Ω_E is complete, α is tangent to s by the convexity of Ω^o. Since the sequences $\{g_i|D^o\}$ and $\{g_i|\alpha\}$ have unbounded differentials, and α is tangent to s, the sequence $\{g_i|s\}$ has unbounded differentials and so does the sequence $\{r_i \circ g_i|s\}$.

Suppose that the sequence $\{r_i \circ g_i(s)\}$ has lengths bounded between ϵ and $\pi - \epsilon$ for $\epsilon > 0$. We can change r_i so that $r_i \circ g_i(s)$ has the same length with $\{r_i\}$ is still bounded. Thus $r_i \circ g_i$ has a fixed point q, the end point, in a distance between ϵ and $\pi - \epsilon$. The associated sequence of eigenvalues satisfies $\{\lambda_i\} \to \infty$. As $\{r_i\}$ is a bounded sequence, the sequence $\{g_i\}$ have eigenvalues $\to \infty$ as $i \to \infty$. Since the norms of eigenvalues

of g_i are always 1, this is a contradiction. Therefore, a subsequence of the lengths of $\{r_i \circ g_i(s)\}$ goes to π or to zero.

In the first case, the sequence of the lengths of $\{g_i(s)\}$ converges to π. This contradicts the proper convexity of $\tilde{\mathcal{O}}$. In the second case, the sequence of the lengths of $\{r_i \circ g_i(s)\}$ converges to 0. Also, the sequence of differential of $r_i \circ g_i$ at $\mathbf{v}_E$ converges to $+\infty$. This also implies the existence of eigenvalues $\to \infty$ as in the above paragraph and we obtain a contradiction.

(iv) Choose an exiting sequence of end neighborhoods U_i and we take convex hulls V_i. V_i is a union of n-simplicies with vertices in U_i. The sequence of the sets of vertices is exiting. Therefore, $\{V_i\}$ is also exiting. One can deform the boundary of V_i by small amounts to make them strictly convex as there are no infinite straight lines in ∂V_i. We can make V_i smooth by taking even smaller smooth neighborhood inside by considering the Zariski closure group action.

(v) Since Ω_E is a complete affine space, $\Omega_E/\mathbf{\Gamma}'_E$ is a complete affine manifold with the norms of eigenvalues holonomy matrices all equal to 1 where $\mathbf{\Gamma}'_E$ denotes the affine transformation group corresponding to $\mathbf{\Gamma}_E$. By D. Fried [26], this implies that $\pi_1(E)$ is virtually nilpotent.

$\square$

3.6. **A complete end is horospherical.** For the results here, there are many overlaps with the results of Crampon-Marquis [25] and Cooper-Long-Tillman [23]. However, the results are somewhat more general than theirs and were originally conceived before their papers appeared. We also make use of Crampon-Marquis [25].

Theorem 3.9. *Let $\mathcal{O}$ be a properly convex n-orbifold with radial ends. Suppose that E is a complete radial end of its universal cover $\tilde{\mathcal{O}}$. Let $\mathbf{v}_E \in \mathbb{S}^n$ be the fixed point of holonomy group $\mathbf{\Gamma}_E$ corresponding to E. Then*

(i) *The eigenvalues of elements of $\mathbf{\Gamma}_E$ have unit norms only.*
(ii) *A finite index subgroup of $\mathbf{\Gamma}_E$ is contained in a unipotent group fixing $\mathbf{v}_E$.*
(iii) *E is horospherical.*
(iv) *There exists an end neighborhood U of E where $\mathrm{bd} U$ is a C^∞-sphere.*
(v) *$\mathbf{\Gamma}_E$ is virtually abelian and a finite index subgroup is in a conjugate of a parabolic subgroup of $\mathsf{SO}(n,1)$ of rank $n-1$ in $\mathsf{SL}_\pm(n+1,\mathbb{R})$ that acts on an ellipsoid in $\mathrm{Cl}(\tilde{\mathcal{O}}) \subset \mathbb{RP}^n$. And hence E has a cusp-type.*

Proof. Note that $\Omega_E/\mathbf{\Gamma}_E$ is a complete affine $(n-1)$-dimensional orbifold for the end domain Ω_E of E.

(i) Since E is complete, Ω_E is identifiable with $\mathbb{R}^{n-1}$. $\mathbf{\Gamma}_E$ induces Γ'_E in $\mathbf{Aff}(\mathbb{R}^{n-1})$ that are of form $x \mapsto Mx + b$ where M is a linear map $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ and b is a vector in $\mathbb{R}^{n-1}$. For each $\gamma \in \mathbf{\Gamma}_E$, we write $\hat{L}(\gamma)$ this linear part.

Suppose that one of the norms of relative eigenvalues of $\hat{L}(\gamma)$ for $\gamma \neq \mathrm{I}$ is greater than 1 or less than 1. At least one eigenvalue of $\hat{L}(\gamma)$ is 1 and $\hat{L}(\gamma)$ restricts to the maximal vector subspace A where all norms of the eigenvalues are 1. Considering γ as a projective transformation fixing the vertex $\mathbf{v}_E$, we see that the eigenvalue of γ corresponding to $\mathbf{v}_E$ equals the ones for the subspace H containing $\mathbf{v}_E$ corresponding to A.

Let this be $\lambda_{\mathbf{v}_E}$. Let H^+ be the open half-space of one dimension higher corresponding to directions in A with $\partial H^+ \ni \mathbf{v}_E$ so that H^+ is invariant under γ.

There exists a projective subspace S of dimension ≥ 0 where the points are associated with eigenvalues λ where $|\lambda| > \lambda_{\mathbf{v}_E}$ up to reselecting γ to be a nonzero integral power of γ if necessary.

Let S' the smallest subspace containing H and S. Let y_1 and y_2 be points of separate components of $U \cap S' - H^+$ so that $\overline{y_1 y_2}$ meets H in its interior.

Then we can choose a subsequence m_i, $m_i \to \infty$, so that $\gamma^{m_i}(y_1) \to f$ and $\gamma^{m_i}(y_2) \to f_-$ as $i \to +\infty$ unto relabeling y_1 and y_2 for a pair of antipodal points $f, f_- \in S$. This implies $f, f_- \in \mathrm{Cl}(\tilde{\mathcal{O}})$, and $\tilde{\mathcal{O}}$ is not properly convex. This is a contradiction. Therefore, the norms of eigenvalues of $\hat{L}(\gamma)$ all equal $\lambda_{\mathbf{v}_E}$ and A is the $(n-1)$-dimensional affine subspace. Thus, the norms of eigenvalues of γ all equal to 1 since the product of the eigenvalues equal ± 1.

(ii) Also, it follows that a finite index subgroup of $\mathbf{\Gamma}_E$ is contained in a unipotent Lie subgroup $\mathcal{N}$. (See the proof of Theorem 3 in D. Fried [26] where we need to generalize the argument to this situation with dimension one higher.)

(iii) The dimension of $\mathcal{N}$ is $n-1$ since a finite index subgroup of $h(\pi_1(E))$ acts cocompactly on $\mathbb{R}^{n-1}$ by the general results of Malcev. (See Chapter II of [49].)

Let U be a component of the inverse image of an end neighborhood so that $\mathbf{v}_E \in \mathrm{bd}(U)$. Since a finite index subgroup F of $\mathbf{\Gamma}_E$ is in $\mathcal{N}$ so that $\mathcal{N}/F$ is compact by Malcev, it follows that $\mathcal{N}$ will act on a smaller open set covering an end neighborhood by taking intersections under images of it under $\mathcal{N}$ if necessary. We let U be this open set from now on. Consequently, $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ is smooth.

We will now show that U is a horospherical end neighborhood: We identify $\mathbf{v}_E$ with $[1, 0, .., 0]$. Let W denote the subspace in $\mathbb{S}^n$ corresponding to ∂A containing $\mathbf{v}_E$. $W \cap \mathrm{Cl}(\tilde{\mathcal{O}})$ is a properly convex subset of W.

Let y be a point of U. Suppose that there exists a sequence $\{g_i \in \mathcal{N}\}$ so that $g_i(y) \to x_0 \in W \cap \mathrm{Cl}(\tilde{\mathcal{O}})$ and $x_0 \neq \mathbf{v}_E$; that is, x_0 in the boundary direction of A. The collection of all such x_0 has a properly convex hull set U_1 in a subspace V in W. The dimension of V is ≥ 0 as it contains x_0.

Now, V is divided into disjoint open hemispheres of various dimensions where $\mathcal{N}$ acts on: By Theorem 3.5.3 of [53], there exists a flag structure $V_0 \subset V_1 \subset \cdots \subset V_k = V$. We take components of complement $V_i - V_{i-1}$. Let $H_V := V - V_{k-1}$.

Suppose that $\dim V = n-1$ for contradiction. Then $H_V \cap U_1$ is not empty since otherwise, we would have a smaller dimensional V. Let h_V be the component of H_V meeting U_1. The action of $\mathcal{N}$ is of dimension $n-1$ and hence locally transitive on H_V: If not, then there exists a one-parameter subgroup $\mathcal{N}'$ that acts trivially on V. Since $\mathcal{N}'$ is not trivial, it acts as a group of nontrivial translations on the affine space H^o. Then $\mathcal{N}'(U)$ is not properly convex. Thus, $\mathcal{N}$ acts transitively on h_V since the orbit of $\mathcal{N}$ in h_V is closed as can be seen easily. Hence, the orbit $\mathcal{N}(y)$ of $\mathcal{N}$ for $y \in H_V \cap U_1$ contains a component of H_V. Since $\mathbf{\Gamma}_E(y) \subset \mathrm{Cl}(\tilde{\mathcal{O}})$ and the convex hull is $\mathcal{N}(y)$ where $\mathcal{N}(y) \subset H_V$. Since $F\mathbf{\Gamma}_E = \mathcal{N}$ for a compact subset F of $\mathcal{N}$, the orbit $\mathbf{\Gamma}_E(y)$ is within a bounded distance from every point of $\mathcal{N}(y)$. Thus, the convex hull is $\mathcal{N}(y)$, and this

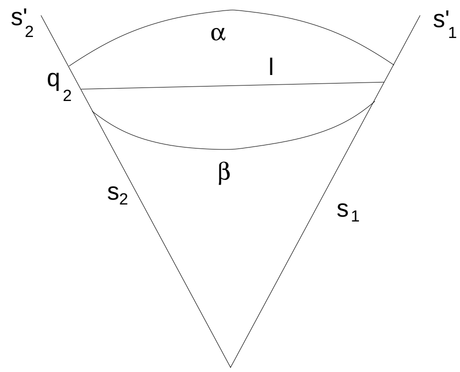

FIGURE 1. The figure for Lemma 3.10.

contradicts the assumption that $\mathrm{Cl}(\tilde{\mathcal{O}})$ is properly convex (compare with arguments in [25].)

Suppose that the dimension of V is $\leq n-2$. Let H be a subspace of dimension 1 bigger than $\dim V$ and containing V and meeting U. Then H is sent to disjoint subspaces or to itself under $\mathcal{N}$. Since $\mathcal{N}$ acts transitively on A, there exists a nilpotent subgroup $\mathcal{N}_H$ of $\mathcal{N}$ acting on H. Now we are reduced to $\dim V$ by one or more. The orbit $\mathcal{N}_H(y)$ for a limit point $y \in H_V$ contains a component of $V - V_{k-1}$ as above. Thus, $\mathcal{N}_H(y)$ contains the same component, an affine subspace. As above, we have a contradiction to the proper convexity.

Therefore, points such as $x_0 \in W \cap \mathrm{bd}(\tilde{\mathcal{O}}) - \{\mathbf{v}_E\}$ do not exist. Hence for any sequence of elements $g_i \in \mathbf{\Gamma}_E$, we have $g_i(y) \to \mathbf{v}_E$.

Hence, $\mathrm{bd} U = \partial U \cup \{\mathbf{v}_E\}$. Since the directions from $\mathbf{v}_E$ to $\mathrm{bd} U \cap \tilde{\mathcal{O}}$ form $\mathbb{R}^{n-1}$, it follows that $\mathrm{bd} U$ is C^1 at $\mathbf{v}_E$. Clearly, $\mathrm{bd} U$ is homeomorphic to an $(n-1)$-sphere. This also proves (iv).

(v) The final item follows by Proposition 7.21 (related to Theorem 1.6) of [25]: By the theorem, we see that $\mathbf{\Gamma}_E$ is in a conjugate of $\mathrm{SO}(n,1)$ and hence acts on an $(n-1)$-dimensional ellipsoid fixing a unique point. This implies that the image group is a Euclidean one acting on some horospheres the ellipsoid given a hyperbolic metric in a neighborhood. Hence, the group is virtually abelian.

□

3.6.1. *The properties of lens-shaped ends.* One of the main result of this section is that a lens-type end has a "concave end neighborhood" that actually covers an end neighborhood.

Given three sequences of points $\{p_i^{(j)}\}$ with $j = 1, 2, 3$ so that $\{p_i^{(j)}\} \to p^{(j)}$ where $p^{(1)}, p^{(2)}, p^{(3)}$ are independent points in $\mathbb{S}^n$. Then a simple matrix computation will show that there exists a bounded sequence $\{r_i\}$ of elements of $\mathbf{Aut}(\mathbb{S}^n)$ so that $r_i(p_i^{(j)}) = p^{(j)}$ for every i and $j = 1, 2, 3$.

A *great segment* is a geodesic segment with antipodal end vertices. It is not properly convex.

A *complete ray* in $\mathbb{S}^n$ with vertex $\mathbf{v}_E$ is a segment with end points $\mathbf{v}_E$ and $\mathbf{v}_{E-}$. Find the tube B_E of complete rays with end points $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ corresponding to elements of Ω_E. The union is a convex domain not properly convex distinguished vertices $\mathbf{v}_E$ and $\mathbf{v}_{E-}$.

We first need the following technical lemmas on recurrent geodesics.

Lemma 3.10. *Let $\mathcal{O}$ be a properly convex real projective n-orbifold with radial ends. Suppose that $g_i \in \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ be a sequence of automorphisms so that $g_i(\mathbf{v}_E) = \mathbf{v}_E$ for a lens-shaped end vertex $\mathbf{v}_E$ and l is a maximal segment in a lens with endpoints in $\mathrm{bd}\tilde{\mathcal{O}}$. (See Figure 1.) Let g_i' denote the induced projective automorphisms on $\mathbb{S}^{n-1}_{\mathbf{v}_E}$. $g_i'(l')$ converges geometrically to l' where l' is the projection of l to the projective link space $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ of $\mathbf{v}_E$. Let P be the 2-dimensional subspace containing $\mathbf{v}_E$ and l. Furthermore, we suppose that*

- *In P, l is in the disk D bounded by two segments s_1 and s_2 from $\mathbf{v}_E$ to and a convex curve α with endpoints q_1 and q_2 in s_1 and s_2 respectively.*
- *β is another convex curve in D with endpoints in s_1 and s_2 so that α and β and parts of s_1 and s_2 bound a convex disk in D.*
- *There is a sequence of points $\tilde{q}_i \in \alpha$ converging to q_1 and $g_i(\tilde{q}_i) \in F$ for a fixed fundamental domain F of $\tilde{\mathcal{O}}$.*
- *The sequences $g_i(D)$, $g_i(\alpha)$, $g_i(\beta)$, $g_i(s_1)$, and $g_i(s_2)$ converge to D, α, β, s_1, and s_2 respectively.*

Then we conclude that

- *If the end points of α and β do not coincide at s_1 or s_2, then α and β must be straight geodesics from q_1 or q_2.*
- *Suppose that the pairs of endpoints of α and β coincide and they are distinct curves. Then there exists no segment in $\mathrm{Cl}(\tilde{\mathcal{O}})$ extending s_1 or s_2 properly.*

Proof. By the geometric convergence conditions, there exists a bounded sequence of elements $r_i \in \mathsf{SL}_{\pm}(n+1, \mathbb{R})$ so that $r_i(g_i(s_1)) = s_1$ and $r_i(g_i(s_2)) = s_2$ and $\{r_i\} \to \mathrm{I}$. Then $r_i \circ g_i$ is represented as an element of $\mathsf{SL}_{\pm}(3, \mathbb{R})$ in the projective plane P containing D. Using $\mathbf{v}_E$ and endpoints of s_1 and s_2 as standard basis, $r_i \circ g_i$ is represented as a diagonal matrix. Moreover $\{r_i \circ g_i(\alpha)\}$ is still converging to α as $\{r_i\} \to \mathrm{I}$. Hence, this implies that the diagonal elements of each $r_i \circ g_i$ are of form λ_i, μ_i, τ_i where $\{\lambda_i\} \to 0, \{\tau_i\} \to +\infty$ as $i \to \infty$ and λ_i is associated with q_1 and μ_i is associated with $\mathbf{v}_E$ and τ_i is associated with q_2.

We have that $\{r_i \circ g_i(\beta)\}$ also converges to β. If the end point of β at s_1 is different from that of α, then $\{\log |\lambda_i/\mu_i|\}$ forms a bounded sequence. In this case, β has to be a geodesic from q_2 since $\{r_i \circ g_i(\beta)\} \to \beta$. And so is α. The similar argument holds for the case involving s_2

For the second item, $\{\mu_i/\tau_i\} \to 0, +\infty$ and $\{\lambda/\mu_i\} \to 0, +\infty$ also since otherwise we can show that β and α have to be geodesic with distinct endpoints as above. If there is a segment s_2' in $\mathrm{Cl}(\tilde{\Omega})$ extending s_2, then $\{r_i \circ g_i(s_2')\}$ converges to a complete ray and so does $\{g_i(s_2')\}$ as $i \to \infty$ or $i \to -\infty$. This contradicts the proper convexity of $\mathcal{O}$.

□

A *concave end-neighborhood* is the open end neighborhood contained in a radial end neighborhood in $\tilde{\mathcal{O}}$ that is a component of a complement of the lens domain in a lens-shaped end neighborhood when the end is a lens-cone over this domain.

Let U be one for the end vertex $\mathbf{v}_E$. If $g \in \mathbf{\Gamma}_E$, then $g(U) = U$ since the lens-part is g-invariant. We will show that the closure of U in $\tilde{\mathcal{O}}$ covers an open end neighborhood in $\tilde{\mathcal{O}}/\mathbf{\Gamma}_E$ and the boundary is a compact hypersurface.

A *totally geodesic hypersurface* in $\tilde{\mathcal{O}}$ is the intersection of a totally geodesic subspace of codimension-one with $\tilde{\mathcal{O}}$.

A *trivial one-dimensional cone* is an open half space in $\mathbb{R}^1$ given by $x > 0$ or $x < 0$.

Recall that if $\pi_1(E)$ is an admissible group, then $\pi_1(E)$ has a finite index subgroup isomorphic to $\mathbb{Z}^{k-1} \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_k$ for some $k \geq 0$ where each $\mathbf{\Gamma}_i$ is hyperbolic or infinite cycle or trivial.

Let us consider Σ_E the real projective $(n-1)$-orbifold associated with E and consider Ω_E as a domain in $\mathbb{S}^{n-1}$ and $h(\pi_1(E))$ induces $\hat{h} : \pi_1(E) \to \mathsf{SL}_\pm(n, \mathbb{R})$ acting on Ω_E. We denote by $\mathrm{bd}\Omega_E$ the boundary of Ω_E in $\mathbb{S}^{n-1}$.

Definition 3.11. A lens-shaped end with the end vertex $\mathbf{v}_E$ is *strictly lens-shaped* if we can choose a lens domain D with the top hypersurfaces A and B so that each ray in $\mathbb{S}^n$ from $\mathbf{v}_E$ in the direction of $\mathrm{bd}\Omega_E$ meets $\mathrm{Cl}(D) - A - B$ at a unique point.

Theorem 3.12. *Let $\mathcal{O}$ be a topologically tame n-orbifold with radial ends and with infinite-index end fundamental groups. Let E be a lens-shaped end of $\tilde{\mathcal{O}}$ associated with an end vertex $\mathbf{v}_E$. Assume that $\pi_1(E)$ is hyperbolic.*

- (i) *The complement of the manifold boundary of the lens-shaped domain D is a nowhere dense set in $\mathrm{bdCl}(D)$ in $\mathbb{S}^n$. Moreover, $\mathrm{bdCl}(D) - \partial D$ is independent of the choice of D. That is, D is strictly lens-shaped. Moreover, each element $g \in \mathbf{\Gamma}_E$ has an attracting fixed point in $\mathrm{bdCl}(D)$ in the ray from $\mathbf{v}_E$ in the direction of $\mathrm{bd}\Omega_E$. The set of attracting fixed points is dense in $\mathrm{bdCl}(D) - A - B$ for the top and the bottom surfaces.*
- (ii) *The closure in $\mathbb{S}^n$ of a concave end-neighborhood of $\mathbf{v}_E$ contains every segment I in $\mathrm{bd}\tilde{\mathcal{O}}$ meeting the closure of a concave end neighborhood of $\mathbf{v}_E$ in I^o. The set $S(\mathbf{v}_E)$ of maximal segments from $\mathbf{v}_E$ in the closure of an end-neighborhood of $\mathbf{v}_E$ is independent of the end-neighborhood, and $\bigcup S(\mathbf{v}_E)$ equals the closure of any end neighborhood of $\mathbf{v}_E$ intersected with $\mathrm{bd}\tilde{\mathcal{O}}$.*
- (iii) *Any concave end neighborhood U of $\mathbf{v}_E$ under the covering map $\tilde{\mathcal{O}} \to \mathcal{O}$ covers the end neighborhood of E of form $U/\pi_1(E)$. That is, a concave end neighborhood is a strict end neighborhood.*
- (iv) *$S(g(\mathbf{v}_E)) = g(S(\mathbf{v}_E))$ for $g \in \pi_1(E)$. Assume that w is the end vertex of an irreducible hyperbolic end. Then $S^o(\mathbf{v}_E) \cap S(w) = \emptyset$ or $\mathbf{v}_E = w$ for end vertices $\mathbf{v}_E$ and w where we defined $S^o(\mathbf{v}_E)$ to denote the relative interior of $\bigcup S(\mathbf{v}_E)$ in $\mathrm{bd}\tilde{\mathcal{O}}$.*

Proof. (i) By Fait 2.12 [3], we obtain that $\pi_1(E)$ is vcf and acts irreducibly on a proper convex cone and the cone has to be strictly convex by Theorem 1.1 of [2].

We have a domain D with boundary components A and B transversal to the radial lines in $R_{\mathbf{v}_E}(\tilde{\mathcal{O}})$. $\mathbf{\Gamma}_E$ acts on both A and B. We have the topological boundary A_1 of A in $\mathrm{Cl}(A)$ and in the topological boundary B_1 of B in $\mathrm{Cl}(B)$.

By Theorem 1.2 of [1], the geodesic flow on $\Omega_E/\mathbf{\Gamma}_E$ is topologically mixing. Thus, each geodesic l in Ω_E, we can find a sequence $\{g_i \in \mathbf{\Gamma}_E\}$ that satisfies the conditions of Lemma 3.10. The two arc in $\mathrm{bd}D$ corresponding to l have endpoints at the identical pair of points. Since this is true for all geodesics, we obtain $A_1 = B_1$ and $A \cup B$ is dense in $\mathrm{bd}D$.

Hence, $\partial D = A \cup B$. We see that $\mathrm{bd}\mathrm{Cl}(D) - \partial D$ is the closures of the attracting and repelling fixed points of $h(\pi_1(E))$ since by Theorem 1.1 of [1], the set of fixed points are dense in $A_1 = B_1$. Therefore this set is independent of the choice of D.

(ii) Consider any segment l in $\mathrm{bd}\tilde{\mathcal{O}}$ with l^o meeting $\mathrm{Cl}(U_1)$ for a concave end-neighborhood U_1 of $\mathbf{v}_E$. This segment is contained in a union of segments from $\mathbf{v}_E$. These segments are all in the boundary of $\mathrm{Cl}(U_1)$ by the fact that the segments of $\mathbf{v}_E$ can only end in the interior of U_1 or correspond to a geodesic in $\mathrm{bd}\Omega_E$. Thus, we may assume without the loss of generality that l is a segment from $\mathbf{v}_E$ in $\mathrm{Cl}(U_1) \cap \mathrm{bd}\tilde{\mathcal{O}}$.

A point of $\mathrm{bd}\Omega_E$ is an end point of a recurrent geodesic since the geodesic flow on E is topologically mixing by Theorem 1.2 [1]. Suppose that the interior of l contains a point p of $\mathrm{bd}\mathrm{Cl}(D) - A - B$ that is in the direction of an end point of a recurrent geodesic m in Ω_E. Lemma 3.10 again applies, and l^o does not meet $\mathrm{bd}\mathrm{Cl}(D) - A - B$.

Given a segment l from $\mathbf{v}_E$ in $\mathrm{bd}\tilde{\mathcal{O}}$ not meeting $\mathrm{bd}\mathrm{Cl}(D) - A - B$ in its interior, the maximal segment l' from $\mathbf{v}_E$ in $\mathrm{bd}\tilde{\mathcal{O}}$ meets $\mathrm{bd}\mathrm{Cl}(D) - A - B$ at the end. Also, $\bigcup S(\mathbf{v}_E)$ is a $\mathbf{\Gamma}_E$-invariant subset.

Let U' be any end-neighborhood associated with $\mathbf{v}_E$. Then since each $g \in \mathbf{\Gamma}_E$ has an attracting fixed point and the repelling fixed point on $\mathrm{bd}\mathrm{Cl}(D) - A - B$, for any segment s in U' from $\mathbf{v}_E$, $\{g^i(s)\}$ converges to an element of $S(\mathbf{v}_E)$. Since the attracting and the repelling fixed points $g \in \mathbf{\Gamma}_E$ is dense in the directions of $\mathrm{bd}\Omega_E$, we have $\bigcup S(\mathbf{v}_E) \subset \mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathcal{O}}$.

We can form $S'(\mathbf{v}_E)$ as the set of maximal segments from $\mathbf{v}_E$ in $\mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathcal{O}}$. Then no segment l in $S'(\mathbf{v}_E)$ has interior points in $\mathrm{bd}\mathrm{Cl}(D) - A - B$ as above. Thus, $S(\mathbf{v}_E) = S'(\mathbf{v}_E)$.

Also, since every points of $\mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathcal{O}}$ has a segment in the direction of $\mathrm{bd}\Omega_E$, it follows that $\bigcup S(\mathbf{v}_E) = \mathrm{Cl}(U') \cap \mathrm{bd}\tilde{\mathcal{O}}$.

We can characterize $S(\mathbf{v}_E)$ as the maximal segments in $\mathrm{bd}\tilde{\mathcal{O}}$ from $\mathbf{v}_E$ ending at points of $\mathrm{bd}\mathrm{Cl}(D) - A - B$. This implies also that $g(S(\mathbf{v}_E)) = S(g(\mathbf{v}_E))$ for any end vertex $\mathbf{v}_E$.

(iii) Given a concave-end neighborhood C_E of an end vertex $\mathbf{v}_E$, we show that

$$g(C_E) = C_E \text{ or } g(C_E) \cap C_E = \emptyset \text{ for } g \in \mathbf{\Gamma} :$$

Suppose that $g(C_E) \cap C_E \neq \emptyset$ and $g(C_E) \neq C_E$. First, we have $g(C_E) \cup C_E \neq \tilde{\mathcal{O}}$: otherwise, we have only two properly convex end vertices in $\tilde{\mathcal{O}}$. Thus, an index-two subgroup of $\mathbf{\Gamma}$ is in $\mathbf{\Gamma}_E$ while we always assume that the index $[\mathbf{\Gamma}, \mathbf{\Gamma}_E]$ is infinite. This is a contradiction.

Since C_E is concave, for each point of $z \in \bigcup S(\mathbf{v}_E)$, there exists a totally geodesic hypersurface D so that a component $C_{E,1}$ of $C_E - D$ is in C_E and $\mathrm{Cl}(C_{E,1}) \ni \mathbf{v}_{C_E}$ for the end vertex $\mathbf{v}_{C_E}$ of C_E. Similar statements hold for $g(C_E)$.

Since $g(C_E) \cap C_E \neq \emptyset$, and $g(C_E) \cup C_E \neq \tilde{\mathcal{O}}$, we have $\mathrm{bd} g(C_E) \cap C_E \neq \emptyset$ or $g(C_E) \cap \mathrm{bd} C_E \neq \emptyset$. Then by above it follows that the interiors of segments in $S(\mathbf{v}_E)$ meet segments of $S(g(\mathbf{v}_E))$ or vice versa. By (ii), this implies that $\bigcup S(\mathbf{v}_E) \subset \bigcup S(g(\mathbf{v}_E))$. Since $\mathbf{v}_E$ is a unique point not in the interior of a segment but in the interior of $\bigcup S(\mathbf{v}_E)$ and similarly for $g(\mathbf{v}_E)$, it follows that $\mathbf{v}_E = g(\mathbf{v}_E)$. Hence, $g \in \mathbf{\Gamma}_E$. Therefore $C_E = g(C_E)$ as C_E is a concave neighborhood. This implies that C_E is a strict end-neighborhood.

(iv) The above implies that $S(\mathbf{v}_E)^o \cap S(w) = \emptyset$ or $\mathbf{v}_E = w$ for end vertices $\mathbf{v}_E$ and w. $\square$

Note that Theorem 3.12 (iii) hold without the hyperbolicity condition on $\pi_1(E)$–(*).

Now we go to the cases when $\pi_1(E)$ has more than two nontrivial factors abelian or hyperbolic. The following theorem shows that lens-shaped ends are totally geodesic. The author obtained the proof of (i-3) from Benoist.

Theorem 3.13. *Let $\mathcal{O}$ be a topologically tame n-orbifold with radial ends and with infinite-index end fundamental groups. Let E be a lens-shaped end of the universal cover $\tilde{\mathcal{O}}$ with the end vertex $\mathbf{v}_E$ and Ω_E the end domain of E. Suppose that the end fundamental group $\mathbf{\Gamma}_E$ is admissible. Then the following statements hold:*

- (i) *For $\mathbb{S}^{n-1}_{\mathbf{v}_E}$, we obtain*
  - (i-1) *Under $\hat{h}(\pi_1(E))$, $\mathbb{R}^n$ splits into $V_1 \oplus \cdots \oplus V_{l_0}$ and Ω_E is the quotient of the sum $C_1 + \cdots + C_{l_0}$ for properly convex or trivial one-dimensional cones $C_i \subset V_i$ for $i = 1, \dots, l_0$*
  - (i-2) *The Zariski closure of a finite index subgroup of $\hat{h}(\pi_1(E))$ is isomorphic to the product $G = G_1 \times \cdots \times G_{l_0} \times \mathbb{R}^{l_0-1}$ where G_i is a reductive subgroup of $\mathbf{Aut}(\mathcal{S}(V_i))$).*
  - (i-3) *Let D_i denote the image of C_i in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$. The number of hyperbolic group factors of $\pi_1(E)$ is $\leq l_0$ and each hyperbolic group factor of $\pi_1(E)$ divides exactly one D_i and acts on other factors trivially.*
  - (i-4) *$\pi_1(E)$ has a rank $l_0 - 1$ free abelian group center corresponding to $\mathbb{Z}^{l_0-1}$ in $\mathbb{R}^{l_0-1}$.*
- (ii) *The end is totally geodesic. D_i correspond to totally geodesic convex domain D_i' of dimension $\dim V_i - 1$ disjoint from $\mathbf{v}_E$.*
- (iii) *g in the center is diagonalizable with positive eigenvalues. For a nonidentity element g in the center, the eigenvalue $\lambda_{\mathbf{v}_E}$ of g at $\mathbf{v}_E$ is strictly between its largest norm and smallest norm eigenvalues.*
- (iv) *The end is strictly lens-shaped and each C_i corresponds to a cone C_i^* over a totally geodesic $(n-1)$-dimensional domain D_i' with $\mathbf{v}_E$. C_i^* contains a concave open invariant set U_i. The end has a neighborhood that is a join of $D_1', .., D_{l_0}'$ with $\mathbf{v}_E$ where the join D' of $D_1', .., D_{l_0}'$ forms the boundary. They are in a lens part of E for any lens-type end neighborhood, and the top and the bottom hypersurfaces of the lens part have the boundary in the boundary of D'.*

(v) $S(\mathbf{v}_E)$ *for nontrivial joined case is equal to the set of maximal segments with vertex* $\mathbf{v}_E$ *in the union* $\bigcup_{i=1}^{j} \mathbf{v}_E * D'_1 * \cdots * \check{D}'_i * \cdots * D'_{l_0}$.

Proof. (i) The proof is in Proposition 1.9.

We will now assume that $\hat{h}(\pi_1(E))$ equals the product without loss of generality.

(ii) Let U be an end neighborhood of E in $\tilde{\mathcal{O}}$. Let $S_1, ..., S_{l_0}$ be the projective subspaces in general position meeting only at the end vertex $\mathbf{v}_E$ where factor groups $\mathbf{\Gamma}_1, ..., \mathbf{\Gamma}_{l_0}$ act irreducibly on. Let C'_i denote the union of segments of length π from $\mathbf{v}_E$ corresponding to the invariant cones in S_i where $\mathbf{\Gamma}_i$ acts irreducibly for each i. The abelian center $\mathbb{Z}^{l_0-1}$ acts as the identity on C_i in the projective space $\mathbb{S}^n_{\mathbf{v}_E}$ of lines through $\mathbf{v}_E$. Let $g \in \mathbb{Z}^{l_0-1}$. Then

- $g|C'_i$ is either an identity or
- $g|C'_i$ fixes a hyperspace $P_i \subset S_i$ not passing through $\mathbf{v}_E$ and g has a representation as a scalar multiplication in the affine subspace $S_i - P_i$ of S_i. Since g commutes with every element of $\mathbf{\Gamma}_i$ acting on C'_i, it follows that $\mathbf{\Gamma}_i$ acts on P_i as well. We let $D'_i = C'_i \cap P_i$.

Lemma 3.14. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. Let E be a lens-shaped end of the universal cover $\tilde{\mathcal{O}}$. Let g be a nontrivial center element of its holonomy group $\mathbf{\Gamma}_E$. Then the largest norm eigenvalue or the smallest norm eigenvalue of g in $\mathbb{S}^n$ is different from the eigenvalue of g at $\mathbf{v}_E$.*

Proof. Let g be an element of $\mathbb{Z}^{l_0-1} - \{0\}$. Suppose that the eigenvalue at $\mathbf{v}_E$ and the largest norm eigenvalue are the same for g without loss of generality since we can choose g^{-1} otherwise.

Let C denote the convex set that is the join of C'_is that have the largest norm eigenvalue of g among all C'_i. Let C^c denote the convex set that is the join of D_i in remaining C'_i. We define a function on the join by considering each point $\vec{x}$ and write it as a unique sum $\vec{x}_1 + \vec{x}_2$ where $\vec{x}_1$ is a vector in the direction of C and $\vec{x}_2$ is a vector in the direction C^c. Let $f_g : C \to \mathbb{S}^1$ defined by $f_g(\mathbf{v}_E) = [0, 1]$ in the homogeneous coordinates of $\mathbb{S}^1$ and the second component of f_g is positive on $C - \{\mathbf{v}_E, \mathbf{v}_{E-}\}$. For given g, $f_g^{-1}([\lambda, 1])$ define a hypersurface in U. It is clear that $f_g \circ g = f_g$ and hence the hypersurface is g-invariant.

We will normalize the coordinates on $\mathbb{S}^1$ so that the second coordinate is 1 always.

We can find $0 < \lambda$ so that $f_g^{-1}([\lambda, 1])$ is disjoint from the given lens-domain D'' since otherwise we can show $\mathbf{v}_E$ is in the limit point of the lens by the action of g^i as $i \to \infty$. Let λ_0 denote the supremum the set of λ satisfying $f_g^{-1}([\lambda, 1])$ is disjoint from the given lens-domain. Let P_0 denote $f_g^{-1}([\lambda_0, 1])$ in U. Then P_0 meets $\mathrm{Cl}(D'') \cap U$ since otherwise we can find the lower supremum by considering that the g-action preserve the values of f_g.

Let $x \in \mathrm{Cl}(D'') \cap U \cap P_0$. Then $g^i(x) \in \mathrm{Cl}(D'') \cap U \cap P_0$ by the g-invariance of the sets. By convexity of the sets, the convex hull C' of $\bigcup_{i \in \mathbb{Z}} g^i(x)$ is a convex set in $\mathrm{Cl}(D'') \cap U \cap P_0$ of dimension ≥ 1. (A simplex with vertices in P_0 is still in P_0 since the inverse image of f_g is totally geodesic.) Then C' is a subset of the boundary of D'' since otherwise there is a lower infimum of the first coordinate of f_g-values. However,

we assume that $\partial D''$ is strictly convex. As the convex disk has nonempty interior, this is impossible.

$\square$

By Lemma 3.14, for all C'_i, all $g \in \mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$ acts as nonidentity. Then the join of all P_i gives us a hyperspace P disjoint from $\mathbf{v}_E$. We will show that it forms a totally geodesic end for E:

From above, we obtain that every nontrivial $g \in \mathbb{Z}^{l_0-1}$ is clearly diagonalizable with positive eigenvalues associated with P_i and $\mathbf{v}_E$ and the eigenvalue at $\mathbf{v}_E$ is different from ones at P_i.

Let us choose C_i. We can find at least one $g' \in \mathbb{Z}^{l_0-1}$ so that g' has the largest norm eigenvalue $\lambda_1(g'_i)$ with respect to C_i as an automorphism of $\mathbb{S}^{n-1}_{\mathbf{v}_E}$. We have $\lambda_1(g') > \lambda_{\mathbf{v}_E}(g')$ by Lemma 3.14.

If $\mathbf{\Gamma}_i$ is a trivial group, then we choose g_i be the identity. Each $C'_i \cap P_i$ has an attracting fixed point of some $g_i \in \mathbf{\Gamma}_i$ restricted to P_i if $\mathbf{\Gamma}_i$ is hyperbolic. We can choose g_i so that the largest norm eigenvalue λ_i of $g_i|P_i$ is sufficiently large. This follows since $\mathbf{\Gamma}_i$ is linear on $S_i - P_i$ where we know that this is true for strictly convex cones by the theories of Koszul and so on. Then by taking k sufficiently large, $g'^k g_i$ has an attracting fixed point in $C'_i \cap P_i$. This must be in $\mathrm{Cl}(\tilde{\mathcal{O}})$. Since the set of attracting fixed points in C'_i is dense in $\partial C'_i \cap P_i$ by Benoist [1], we obtain $C'_i \cap P_i \subset \mathrm{Cl}(\tilde{\mathcal{O}})$.

Let D'_i denote $C'_i \cap P_i$. Then the join D' of $\mathrm{Cl}(D'_1), .., \mathrm{Cl}(D'_{l_0})$ equals $P \cap \mathrm{Cl}(\tilde{\mathcal{O}})$, which is $h(\pi_1(E))$-invariant. And D'^o is a properly convex subset. If any point of D^o is in $\mathrm{bd}\tilde{\mathcal{O}}$, then D' is a subset of $\mathrm{bd}\tilde{\mathcal{O}}$ by convexity; and it is not possible to find a lens for the end. Therefore, $D'^o \subset \tilde{\mathcal{O}}$.

(iii) If the eigenvalue at $\mathbf{v}_E$ of nontrivial $g \in \mathbb{Z}^{l_0-1}$ is the largest norm or smallest norm eigenvalue of multiplicity at least two, then we have S_i where g acts as the identity, which was ruled out by Lemma 3.14. If the eigenvalue at $\mathbf{v}_E$ of an element $g \in \mathbb{Z}^{l_0-1}$ is the largest or smallest ones of multiplicity one, we can find an open segment s from $\mathbf{v}_E$ meeting D'^o in $\tilde{\mathcal{O}}$ in the interior s^o. Acting by $g^i(s)$ for $i \in \mathbb{Z}$, we obtain a segment of length π in $\mathrm{Cl}(\tilde{\mathcal{O}})$. This contradicts the proper convexity.

(iv) Let P be the minimal totally geodesic subspace containing all of $P_1, \dots, P_{l_0}$. The hyperspace P separates $\tilde{\mathcal{O}}$ into two parts, ones in the end neighborhood U and the subspace outside it. Clearly U covers Σ_E times an interval by the action of $h(\pi_1(E))$ and the boundary of U goes to a compact orbifold projectively diffeomorphic to Σ_E.

Recall D'_i the set $C'_i \cap P_i$ for each $i = 1, \dots, l_0$. The action of $\pi_1(E)$ on geodesics on Ω_E is ergodic since it is a virtual product of the hyperbolic groups or trivial groups and the abelian group in the center that acts ergodically on the geodesics in D^o: Let ρ denote a geodesic passing the interior of Ω_E. where an end point p_1 is in the join J_1 of $\mathrm{Cl}(D_{i_1}) * \cdots * \mathrm{Cl}(D_{i_{l_0}})$ and and the other end point p_2 has to be in the join J_2 of the remaining D_is. We can choose $g_i \in \mathbb{Z}^{l_0-1}$ so that each of the sequences $g_i|J_1$ and $g_i|J_2$ converges to identities. In this case, we see that $g_i(\rho)$ approximates ρ arbitrarily as $g_i \to \infty$ in $\mathbf{\Gamma}_E$. (This exists as $\mathbb{Z}^{l_0-1}$ acts cocompactly on a properly convex l_0-simplex Δ in a projective space with transferred eigenvalues from the joins. This follows since in lattice in $\mathbb{R}^{l_0-1}$ any vector is approximated by rational sum of basis vectors.)

The above shows that any geodesic ρ in Ω_E is recurrent. As in the proof of Theorem 3.12 (i), we use this recurrency in Ω_E to show that E is strictly lens-shaped.

Since each $g \in \mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$ has $\mathbf{v}_E$ with the eigenvalue strictly between the largest norm and the smallest norm ones, it follows that each point of ∂D can be seen as a limit point of some sequence $g_i(x)$ for $x \in U$. Therefore ∂D is exactly the boundary of the top hypersurface and the bottom one.

(v) This follows by considering each irreducible part of the lens.

$\square$

3.7. **The openness of lens properties.** A *radial affine connection* is an affine connection on $\mathbb{R}^{n+1} - \{O\}$ invariant under the radial dilatation $S_t : \vec{v} \to t\vec{v}$ for every $t > 0$.

Theorem 3.15. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. Let E be a properly convex end of the universal cover $\tilde{\mathcal{O}}$. Let $\mathrm{Hom}(\pi_1(E), \mathrm{SL}_\pm(n+1, \mathbb{R}))$ be the space of representations of the fundamental group of an n-orbifold Σ_E with an admissible fundamental group. Then*

(i) *the lens-shapedness is equivalent to the strict lens-shapedness, and*
(ii) *the subspace of lens-shaped representations is open.*

Proof. (i) If $\pi_1(E)$ is hyperbolic, then the equivalence is in Theorem 3.12 (i) and if $\pi_1(E)$ is a virtual product of hyperbolic groups and abelian groups, then we use Theorem 3.13 (iv).

(ii) Let μ be a representation $\pi_1(E) \to \mathrm{SL}_\pm(n+1, \mathbb{R})$ acting on a strictly convex n-domain K bounded by two open n-cells A and B and $\mathrm{bd}K - A - B$ is a nowhere dense set. We assume that A and B are smooth and convex. We note that $K/\mu(\pi_1(E))$ is a compact manifold with boundary with two closed n-orbifold components $A/\mu(\pi_1(E)) \cup B/\mu(\pi_1(E))$. We see that A and B are strictly convex hypersurfaces. By using the theory of deformations of geometric structures on compact orbifolds, we obtain a manifold N' diffeomorphic to $K/\mu(\pi_1(E))$. $\tilde{N}'$ is a manifold with two boundary components A' and B' and developing into $\mathbb{S}^{n-1}$. Suppose that μ' is sufficiently near μ. Then μ' must act on A' and B' sufficiently near in the compact open C^1-topology.

Given K, we can find a convex $(n+1)$-domain $K' \subset K^o$ bounded by two smooth open n-cells A' and B' in K^o. We may also assume that K' is strictly convex.

Since K is properly convex, we choose K' as above. The linear cone $C(K) \subset \mathbb{R}^{n+1}$ over K has a smooth strictly convex hessian function V by Vey's work [55]. Let $C(K')$ denote the linear cone over K'. For the fundamental domain F of $C(K')$ under the action of $\mu(\pi_1(E))$ extended by a transformation $\gamma : \vec{v} \mapsto 2\vec{v}$, the hessian restricted to $F \cap C(K')$ has a lower bound. Also, the boundary $\partial C(K')$ is strictly convex in any affine coordinates in any transversal subspace to the radial directions at any point.

Let M be $C(K')/\langle \mu(\pi_1(E)), \gamma \rangle$, a compact orbifold. Note that S_t, $t \in \mathbb{R}^+$, becomes an action of circle on M. The change of representation μ to $\mu' : \pi_1(E) \to \mathbf{Aut}(\mathbb{S}^n)_{\mathbf{v}_E}$ is realized by a change of holonomy representations of M and hence the change of affine connections on $C(K)$. Since S_t commutes with image of μ and μ', S_t still gives us a circle action on M with a different affine connection. We may assume without loss of generality that the circle action is fixed and M is invariant under this action.

If we change $C(K')$ to to a cone $C(K'')$ of K'' by a sufficiently small change in the radial affine connection which does not change the radial directions locally, the positive definiteness of the hessian in the fundamental domain and the boundary transversal strict convexity is preserved. Thus K'' is also a properly convex domain by Koszul's work [42].

Thus the perturbed K'' is a properly convex domain with strictly convex boundary A'' and B''. The complement $\Lambda = \mathsf{Cl}(K'') - A'' - B''$ is a closed subset. Then by Theorems 3.12 and 3.13, we obtain that the end is also strictly lens-shaped. $\square$

4. The characterization of lens-shaped representations

The main purpose of this section is to characterize the lens-shaped representations in terms of eigenvalues and try to describe this space.

First, we prove the inequality result for irreducible cases of ends. Next, we give the definition of uniform middle-eigenvalue conditions. We show that the uniform middle-eigenvalue conditions imply the distanced nature of the action. Finally, we show the equivalence of the lens condition and the uniform middle-eigenvalue condition in Theorem 4.5. We also show the openness of the uniform middle-eigenvalue conditions.

4.1. **The uniform middle-eigenvalue condition and the openness.** Let $\mathcal{O}$ be a properly convex real projective orbifold with radial ends and $\tilde{\mathcal{O}}$ be the universal cover. Let E be an end of $\tilde{\mathcal{O}}$ and $\mathbf{v}_E$ be the end vertex. Let $h : \pi_1(E) \to \mathsf{SL}_{\pm}(n+1,\mathbb{R})_{\mathbf{v}_E}$ be a homomorphism and suppose that $\pi_1(E)$ is hyperbolic. Assume that for each nonidentity central element of $\pi_1(E)$, the eigenvalue of g at the vertex $\mathbf{v}_E$ of E has a norm strictly between the maximal and the minimal norms of eigenvalues of g —(*). We say that h satisfies the *middle-eigenvalue condition*. Recall $\mathcal{L}_1$ from the beginning of Section 3. We denote by $\hat{h} : \pi_1(E) \to \mathsf{SL}_{\pm}(n,\mathbb{R})$ the homomorphism $\mathcal{L}_1 \circ h$. Since $\hat{h}$ is a holonomy of a closed convex real projective $(n-1)$-orbifold, and Σ_E is assumed to be properly convex, $\hat{h}(\pi_1(E))$ divides a properly convex domain Ω_E in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$.

We denote by $\tilde{\lambda}_1(g), ..., \tilde{\lambda}_n(g)$ the norms of eigenvalues of $\hat{h}(g)$ so that

$$\tilde{\lambda}_1(g) \geq ... \geq \tilde{\lambda}_n(g)$$

holds. These are called the *relative norms of eigenvalues* of g. We denote by the norms of eigenvalues of g by $\lambda_1(g), ..., \lambda_n(g), \lambda_{\mathbf{v}_E}(g)$ where $\lambda_i(g) = \tilde{\lambda}_i(g)/\lambda_{\mathbf{v}_E}(g)^{1/n}$ for $i = 1, .., n$. Since $\mathbf{v}_E$ is fixed, the norm of the corresponding eigenvalue $\lambda_{\mathbf{v}_E}(g)$ is one of these.

Note here that $\lambda_1(g), \tilde{\lambda}_1(g), \lambda_n(g), \tilde{\lambda}_n(g), \lambda_{\mathbf{v}_E}(g)$ are all positive, mainly by the work of Benoist (see [6]), which really goes back to Kuiper, Koszul, and so on.

We define the $\mathsf{length}(g)$ to be $\log(\tilde{\lambda}_1(g)/\tilde{\lambda}_n(g)) = \log(\lambda_1(g)/\lambda_n(g))$, which is the infimum of the Hilbert metric lengths of the associated closed curves in $\Omega_E/\hat{h}(\pi_1(E))$.

We also recall the work of Guichard [34] following the initial work by Benoist[1]. Each element $g \in \mathsf{SL}_{\pm}(n+1,\mathbb{R})$

- that has the largest and smallest norms of the eigenvalues which are distinct and
- the largest or the smallest norm correspond to the unique eigenvectors

is said to be *biproxisimal*. All nonfinite order elements of $\hat{h}(\pi_1(E))$ are biproximal and there is a finite index subgroup of biproximal elements and the identity. Note also an element is *proximal* if and only if it is biproximal when Γ is a hyperbolic group.

Suppose that $g \in \mathbf{\Gamma}_E$ is proximal. We define

$$\alpha_g := \frac{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_n(g)}{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_{n-1}(g)}, \beta_g := \frac{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_n(g)}{\log \tilde{\lambda}_1(g) - \log \tilde{\lambda}_2(g)}, \tag{5}$$

and denote by $\mathbf{\Gamma}_E^p$ the set of proximal elements. We define

$$\beta_{\mathbf{\Gamma}_E} := \sup_{g \in \mathbf{\Gamma}_E^p} \beta_g, \alpha_{\mathbf{\Gamma}_E} := \inf_{g \in \mathbf{\Gamma}_E^p} \alpha_g.$$

Proposition 20 of [34] shows that we have

$$1 < \alpha_{\Omega_E} \leq \alpha_\Gamma \leq 2 \leq \beta_\Gamma \leq \beta_{\Omega_E} < \infty \tag{6}$$

for constants α_{Ω_E} and β_{Ω_E} depending only on Ω_E since Ω_E is properly and strictly convex.

Here, it is easy to see that $\alpha_{\mathbf{\Gamma}_E}, \beta_{\mathbf{\Gamma}_E}$ depends on $\hat{h}$ and form a function of convex divisible part of $\mathsf{Hom}(\pi_1(E), \mathsf{SL}_\pm(n+1, \mathbb{R}))/\mathsf{SL}_\pm(n+1, \mathbb{R})$ with algebraic convergence topology.

Theorem 4.1. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. Let E be a properly convex end of the universal cover $\tilde{\mathcal{O}}$. Let $\mathbf{\Gamma}_E$ be a hyperbolic group or an abelian group. Then there exists a constant $C > 0$ depending on the representation*

$$C^{-1}\mathsf{length}(g) \leq \mathsf{clw(g)} \leq \mathsf{Clength(g), g} \in \mathbf{\Gamma}_\mathsf{E} - \{\mathsf{I}\},$$

and there exists a constant $J > 0$ such that

$$J^{-1}\mathsf{clw}(g) \leq \log \tilde{\lambda}_1(g) \leq J\mathsf{clw}(g)$$

for every proximal element $g \in \hat{h}(\pi_1(E))$ where J depends only on the representation $\hat{h}$.

Proof. First assume that $\mathbf{\Gamma}_E$ is hyperbolic. The first inequality of the first item comes from the fact that any closed loop following the conjugated words of g can be shorted to a closed geodesic. The second inequality follows from the fact that one can choose a conjugate word following the closed geodesic.

Since there is a biproximal subgroup of finite index, we concentrate on biproximal elements only. We obtain from above that

$$\frac{\log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)}}{\log \frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_2(g)}} \leq \beta_{\Omega_E}.$$

We deduce that

$$\frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_2(g)} \geq \left(\frac{\lambda_1(g)}{\lambda_n(g)}\right)^{1/\beta_{\Omega_E}} = \left(\frac{\tilde{\lambda}_1(g)}{\tilde{\lambda}_n(g)}\right)^{1/\beta_\Omega} = \exp(\frac{\mathsf{length}(g)}{\beta_{\Omega_E}}). \tag{7}$$

Since we have $\tilde\lambda_i \leq \tilde\lambda_2$ for $i \geq 2$, we obtain

$$\frac{\tilde\lambda_1(g)}{\tilde\lambda_i(g)} \geq \left(\frac{\lambda_1}{\lambda_n}\right)^{1/\beta_{\Omega_E}} \tag{8}$$

and since $\tilde\lambda_1 \cdots \tilde\lambda_n = 1$, we have

$$\tilde\lambda_1(g)^n = \frac{\tilde\lambda_1(g)}{\tilde\lambda_2(g)} \cdots \frac{\tilde\lambda_1(g)}{\tilde\lambda_{n-1}(g)} \frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)} \geq \left(\frac{\tilde\lambda_1(g)}{\tilde\lambda_n(g)}\right)^{\frac{n-2}{\beta}+1}.$$

We obtain

$$\log \tilde\lambda_1(g) \geq \frac{1}{n}\left(1 + \frac{n-2}{\beta_{\mathbf{\Gamma}_E}}\right) \mathrm{length}(g). \tag{9}$$

By similar reasoning, we also obtain

$$\log \tilde\lambda_1(g) \leq \frac{1}{n}\left(1 + \frac{n-2}{\alpha_{\mathbf{\Gamma}_E}}\right) \mathrm{length}(g). \tag{10}$$

Now, if $\mathbf{\Gamma}_E$ is abelian, then $\mathbf{\Gamma}_E$ is diagonalizable and this result is elementary. □

We remark that the above holds for hyperbolic manifolds where $\lambda_2 = \cdots = \lambda_{n-2} = 1$ and the equations 9 and 10 are equalities in this cases with $\alpha_{\mathbf{\Gamma}_E} = 2 = \beta_{\mathbf{\Gamma}_E}$.

Again let $\lambda_1(g)$ denote the largest norm eigenvalue of g. If $\log \frac{\lambda_1(g)}{\lambda_{\mathbf{v}_E}(g)} > 0$ or equivalently $\tilde\lambda_1(g)/\lambda_{\mathbf{v}_E}(g)^{1/n} > \lambda_{\mathbf{v}_E}(g)$ for every $g \in \mathbf{\Gamma}_E - \{\mathrm{I}\}$, then $\mathbf{\Gamma}_E$ is said to satisfy the *middle-eigenvalue condition*.

Recall the Vey decomposition, we may assume without loss of generality that $\mathbf{\Gamma}_E = \mathbb{Z}^{l_0-1} \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_{l_0}$ where $\mathbf{\Gamma}_i$ is an irreducible factor for each i, which corresponds to the decomposition

$$C_1 \oplus \cdots \oplus C_{l_0}$$

where Γ_i acts on $\mathcal{P}(C_i)$ irreducibly. We assume that each element $g \in \mathbf{\Gamma}_i$ has the largest norm eigenvalue $\lambda_1(g)$ with the vector occurring on C_i. Suppose that there exists a constant $J > 0$ independent of $g \in \mathbf{\Gamma}_i - \{\mathrm{I}\}$ for each i or $g \in \mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$ such that

$$J^{-1}\mathrm{length}(g) \leq \log \frac{\lambda_1(g)}{\lambda_{\mathbf{v}_E}(g)} \leq J\mathrm{length}(g)$$

where if $\mathbf{\Gamma}_i$ is trivial, we do not require anything. Then we say that h satisfies the *uniform middle-eigenvalue condition*.

A *unit tangent bundle* $U\Omega_E$ on a manifold Ω_E is the equivalence class of tangent vectors of Ω_E given by $\vec v \sim \vec w$ if $\vec v = s\vec w$ for $s > 0$. The *unit tangent bundle* $U\Sigma_E$ of Σ_E is given as the $U\Omega_E/\Gamma$ where the deck transformation group Γ acts by

$$g(x, [\vec v]) \mapsto (g(x), [Dg(\vec v)]) \text{ for } x \in \Omega_E, [\vec v] \in U\Omega_E, g \in \Gamma$$

where Dg is the differential of g. There exists a projective geodesic flow

$$\Phi : U\Sigma_E \times \mathbb{R} \to U$$

given by $\Phi(t, x)$ is a projective geodesic line of length t with respect to some metric on $U\Sigma_E$.

Let $U\Sigma_E$ denote the unit tangent bundle on the orbifold Σ_E. A *geodesic current* is a Borel probability measure on $U\Sigma_E$ invariant under the projective geodesic flow.

Let E be irreducible to begin with. Let $\mathcal{C}(\Sigma_E)$ denote the set of geodesic currents with the weak *-topology (see Lemma 1.2 of [32] for example). Let $\mathcal{C}_1(\Sigma_E)$ denote the set of normalized geodesic currents of Σ_E, which is a compact subset. Then $\mathcal{C}_1(\Sigma_E)$ is isomorphic to $\mathcal{C}(\Sigma_E) - \{O\}/\sim$ where $\mu \sim \nu$ if $\mu = s\nu$ for $s > 0$. Let $\mathcal{S}(\Sigma_E)$ denote the countable set of currents supported on a finite union of closed geodesics with rational weights. Let $\mathcal{S}_1(\Sigma_E)$ denote the countable subset of $\mathcal{C}_1(\Sigma_E)$ supported on a finite union of closed geodesics with rational weights. These are countable dense subsets of $\mathcal{C}(\Sigma_E)$ and $\mathcal{C}_1(\Sigma_E)$ respectively if Γ is hyperbolic by the work of [1].

Every element $g \in \mathbf{\Gamma}_E$ is written as

$$g_1 \cdots g_{l_0}\zeta$$

for $\zeta \in \mathbb{Z}^{l_0-1}$ and $g_i \in \mathbf{\Gamma}_i$, $i = 1, \dots, l_0$. We call this the standard form. We define a function $\Lambda_1 : \mathcal{C}(E) \to \mathbb{R}$ by the following:

- First define $\Lambda_1(c)$ for a closed curve to be $\min \Lambda_g$ where

$$\Lambda_g := \left\{ \log \frac{\lambda_1(g_1)}{\lambda_{\mathsf{v}_E}(g_1)}, \dots, \log \frac{\lambda_1(g_{l_0})}{\lambda_{\mathsf{v}_E}(g_{l_0})}, \log \frac{\lambda_1(\zeta)}{\lambda_{\mathsf{v}_E}(\zeta)} \right\}$$

  for an element g representing c where $g = g_1 \cdots g_{l_0}\zeta$ for $g_i \in \Gamma_i - \{\mathrm{I}\}$ and $\zeta \in \mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$.
- If c is a disjoint union of closed curve $c_1, \dots, c_j$ for some $j \geq 1$, then we define

$$\Lambda_1(c) := \Lambda_1(c_1) + \cdots + \Lambda_1(c_j).$$

  If we multiply c by a constant $s, s > 0$, then $\Lambda_1(sc) = s\Lambda_1(c)$.
- For an arbitrary current c, we let c_i be a sequence of currents supported on unions of closed curves converging to c under the weak $*$-topology. Then define $\Lambda_1(c)$ to be

$$\inf_{\{c_i\}} \liminf_i \left\{ \Lambda_1(c_i) | \{c_i\} \text{ is a sequence of currents converging to } c \right\}.$$

  This defines a map $\Lambda_1 : \mathcal{C}(E) \to \mathbb{R}^+ \cup \{0\}$.

(Here, we allow smooth closed curves as well as geodesics for geometric reasons of the metrics.)

If h satisfies the uniform middle-eigenvalue condition, then we have $\Lambda_1(c) > J$ for a constant $J > 0$ for every $c \in \mathcal{C}_1(E)$. This is true when $\mu = 0$ identically.

We define $\mathrm{Hom}_{\mathsf{v}_E,\hat{h}}(\pi_1(E), \mathrm{SL}_\pm(n+1,\mathbb{R}))$ to be the subspace where the linear part is fixed to be $\hat{h} : \pi_1(E) \to \mathrm{SL}_\pm(n,\mathbb{R})$.

Proposition 4.2. *Let $\mathcal{O}$ be a properly convex real projective manifold with radial ends or totally geodesic ends. We are given a properly convex end E of the universal cover $\tilde{\mathcal{O}}$. Then the subset of $\mathrm{Hom}_{\mathsf{v}_E,\hat{h}}(\pi_1(E), \mathrm{SL}_\pm(n+1,\mathbb{R}))$ consisting of representations satisfying the uniform middle-eigenvalue condition is open.*

Proof. Let $h : \pi_1(E) \to \mathrm{SL}_\pm(n+1,\mathbb{R})$ be the homomorphism with the linear part $\hat{h}$.

For element μ^h of $H^1(\pi_1(E), \mathbb{R})$, we define a function $\mu^h_C : \mathcal{C}(E) \to \mathbb{R}$ given by $\mu^h_C(c) = \mu^h(g)$ for the element $g \in \Gamma$ representing c. We extend as above. We note that $\lambda^h_{\mathbf{v}_E}(g)$ equals $\exp \mu^h(g)$ for some μ^h.

Note that $g \in \pi_1(E) \to \log \lambda^h_{\mathbf{v}_E}(g)$ is an element μ^h of $H^1(\pi_1(E), \mathbb{R})$ so that we have for some constant $C_{\mu^h} > 0$ independent of g but depending on μ^h

$$-C_{\mu^h}\mathrm{length}(g) \leq \log \lambda^h_{\mathbf{v}_E}(g) \leq C_{\mu^h}\mathrm{length}(g), g \in \pi_1(E).$$

We have $\lambda^h_1(g) = \tilde\lambda^h_1(g)/(\lambda^h_{\mathbf{v}_E}(g))^{\frac{1}{n}}$ where $\lambda^h_1(g)$ is the largest norm of the relative eigenvalue of $h(g)$ and so we have

$$\log\left(\frac{\lambda^h_1(g)}{\lambda^h_{\mathbf{v}_E}(g)}\right) = \log \tilde\lambda^h_1(g) - \frac{n+1}{n}\log \lambda^h_{\mathbf{v}_E}(g).$$

We define $\tilde\Lambda^h_1 : \mathcal{C}(E) \to \mathbb{R}^+ \cup \{0\}$ by taking $\tilde\Lambda^h_1(c)$ be $\tilde\lambda^h_1(g)$ for the element $g \in \Gamma$ representing c. We extend as above. This is a lower-semicontinuous function. Then we obtain the image of the graph of $\tilde\Lambda^h_1$ in $\mathcal{C}_1(E) \times \mathbb{R}^+ \cup \{0\}$. Note that this set is bounded below by a positive constant map by Theorem 4.1. This is a bounded set. We take the closure and take the boundary of the set $\tilde L(h)$ of the open set of points below the graph. The uniform middle-eigenvalue condition is equivalent to the condition that the graph of $\exp \frac{n+1}{n}\mu^h : \mathcal{C}_1(E) \to \mathbb{R}^+ \cup \{0\}$ lies strictly below $\tilde L(h)$.

Now the condition of the graph of $\exp \frac{n+1}{n}\mu^h$ being below $\tilde L^h$ is an open condition by the lower-semicontinuity of $\tilde L^h$.

Since for each factor the openness holds, the openness of the join will hold also. □

We cannot show that the middle-eigenvalue condition does not imply the uniform middle-eigenvalue condition. This could be false. For example, there could be a sequence of elements $g_i \in \Gamma$ so that $\lambda_1(g_i)/\lambda_{\mathbf{v}_E}(g_i) \to 1$ while Γ satisfies the middle-eigenvalue condition. Certainly, we could have an element g where $\lambda_1(g) = \lambda_{\mathbf{v}_E}(g)$. However, even if there is no such element, we might still have a counter-example. For example, suppose that we might have

$$\frac{\log(\lambda_1(g_i)/\lambda_{\mathbf{v}_E}(g_i))}{\mathrm{length}(g)} \to 0.$$

(Such assignments are not really understood but see Benoist [6]. Also, an analogous phenomenon seems to happen with the Margulis space-time and Margulis invariants.)

4.2. **The uniform middle-eigenvalue condition and the orbits.** Let E be an end of the universal cover $\tilde{\mathcal{O}}$ of a properly convex real projective orbifold $\mathcal{O}$ with radial ends. In this subsection, we assume that $\mathbf{\Gamma}_E$ is irreducible and hyperbolic. Assume that $\mathbf{\Gamma}_E$ satisfies the uniform middle-eigenvalue condtion. There exists a $\mathbf{\Gamma}_E$-invariant convex set K distanced from $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$ by Theorem 3.5. Then for the corresponding tube $\mathcal{T}_{\mathbf{v}_E}$, $\partial\mathcal{T}_{\mathbf{v}_E} \cap K$ is a compact subset distanced from $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$. Let K^* be the boundary component of this set closer to $\mathbf{v}_E$. Then K^* meets every segment in $\partial\mathcal{T}_{\mathbf{v}_E}$ from $\mathbf{v}_E$ to $\mathbf{v}_{E-}$. Let C_1 denote the convex hull of K^*. Then C_1 is a $\mathbf{\Gamma}_E$-invariant subset of $\mathcal{T}_{\mathbf{v}_E}$.

Also, it follows that $K^* \cap \partial \mathcal{T}_{\mathbf{v}_E}$ contains all attracting and repelling fixed points of $\gamma \in \mathbf{\Gamma}_E$ by invariance and the middle-eigenvalue condition.

Lemma 4.3. *Suppose that γ_i is a sequence of elements of $\mathbf{\Gamma}_E$ acting on $\mathcal{T}_{\mathbf{v}_E}$. The sequence of attracting fixed points a_i and the sequence of repelling fixed points b_i are so that $a_i \to a_\infty$ and $b_i \to b_\infty$ where a_∞, b_∞ are not in $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$ for $a_\infty \neq b_\infty$. Suppose that the sequence $\{\lambda_i\}$ of eigenvalues where λ_i corresponds to a_i converges to $+\infty$. Then for*

$$M := \mathcal{T}_{\mathbf{v}_E} - \bigcup_{i=1}^{\infty} \overline{b_i \mathbf{v}_E} \cup \overline{b_i \mathbf{v}_{E-}},$$

the point a_∞ is the limit of $\{\gamma_i(x)\}$ for any $x \in M$.

Proof. There exists a totally geodesic sphere $\mathbb{S}^{n-1}_i$ at b_i supporting $\mathcal{T}_{\mathbf{v}_E}$. a is uniformly bounded away from $\mathbb{S}^{n-1}_i$ for i sufficiently large. $\mathbb{S}^{n-1}_i$ bounds an open hemisphere H_i containing a_i where a_i is the attracting fixed point so that there exists a euclidean metric $d_{E,i}$ so that $\gamma_i | H_i : H_i \to H_i$ is a contraction by the inverse of the factor

$$\max \left\{ \frac{\tilde\lambda_1(\gamma_i)}{|\tilde\lambda_2(\gamma_i)|}, \frac{\tilde\lambda_1(\gamma_i)}{\lambda_{\mathbf{v}_E}(\gamma_i)^{\frac{n+1}{n}}} \right\}.$$

Note that $\{\mathrm{Cl}(H_i)\}$ converges geometrically to $\mathrm{Cl}(H)$ for an open hemisphere containing a in the interior. (Actually, we can choose $d_{E,i}$ so that $\{d_{E,i}\}$ is uniformly convergent for any compact subset J of H_∞.) Equation 8 and the fact that $\Lambda_1(g_i)/\mathrm{length}(g_i) > C_0$ for any uniform $C_0 > 0$ imply that the sequence of factors converges to ∞. Since any compact subset J of M the distance $\mathbf{d}(J, \partial H_i)$ is uniformly bounded by a constant C_1, it follows that $\{g_i(J)\}$ geometrically converges to a. ☐

Proposition 4.4. *Let $z \in \mathcal{T}^o_{\mathbf{v}_E}$. By the uniform middle-eigenvalue condition, the sequences of orbit elements of z limit to points of K^* only.*

Proof. Given $z \in \mathcal{T}^o_{\mathbf{v}_E}$, we let γ_i be any sequence in $\mathbf{\Gamma}_E$ so that the corresponding sequence of $\gamma_i(z)$ in Ω_E converges to a point z' in $\mathrm{bd}\Omega_E$. Let z_∞ denote the point of K^* corresponding to z' in K^*.

The action of $\mathbf{\Gamma}_E$ on $\mathrm{bd}\Omega_E$ is a convergence group. Then we can assume that for the attracting fixed points a_i and r_i of γ_i, we have $\{a_i\} \to a$ and $\{r_i\} \to r$ for $a_i, r_i, a, r \in K^*$. Assume $a \neq r$ first. By Lemma 4.3, we have $\{\gamma_i(z)\} \to a$ and hence $z_\infty = a$.

However, it could be that $a = r$. In this case, we choose $\gamma_0 \in \mathbf{\Gamma}_E$ so that $\gamma_0(a) \neq r$. Then $\gamma_0\gamma_i$ has the attracting fixed point a'_i so that we obtain $\{a'_i\} \to \gamma_0(a)$ and repelling fixed points r'_i so that $\{r'_i\} \to a$ holds. (These are standard facts for convergence group actions.)

Then as above $\{\gamma_0\gamma_i(z)\} \to \gamma_0(a)$. and we need to multiply by γ_0^{-1} now to show $\{\gamma_i(z)\} \to a$.

☐

4.3. **The uniform middle-eigenvalue condition and the lens-shaped ends.** A *radially foliated end neighborhood system* of $\mathcal{O}$ is a collection of end neighborhoods of $\mathcal{O}$ that is radially foliated where each radial ray from the end vertex meets the boundary

of the end neighborhoods uniquely and the complement is a compact suborbifold with the boundary the union of boundary components of the end neighborhoods.

We say that $\mathcal{O}$ satisfies the *triangle condition* if for $\tilde{\mathcal{O}}$, the interior of every triangle T with ∂T in $\mathrm{bd}\tilde{\mathcal{O}}$ is a subset of an end neighborhood U in $\tilde{\mathcal{O}}$ from a fixed radially foliated end neighborhood system of $\mathcal{O}$.

In [13], we will show that this condition is satisfied if $\pi_1(\mathcal{O})$ is relatively hyperbolic with respect to the end fundamental groups. We will prove this in [13] since it is a global result and not a result on ends only.

Theorem 4.5. *Let $\mathcal{O}$ a topologically tame properly convex real projective orbifold with radial ends and with admissible end fundamental groups. Assume that $\mathcal{O}$ satisfies the triangle condtion. Let $\mathbf{\Gamma}_E$ be the holonomy group of a properly convex end E of a properly convex real projective orbifold with radial ends. Then the following statements are equivalent* :

(i) $\mathbf{\Gamma}_E$ *is of lens-type.*
(ii) $\mathbf{\Gamma}_E$ *satisfies the uniform middle-eigenvalue condition.*

Proof. By Lemma 4.6, every triangle T with ∂T in $\mathrm{bd}\tilde{\mathcal{O}}$ that is a subset of an end neighborhood in $\tilde{\mathcal{O}}$ in an end neighborhood system does not have the corresponding end vertex as its vertex.

(ii) $\Rightarrow$ (i): First assume that $\mathbf{\Gamma}_E$ is irreducible hyperbolic. Let $\partial\mathcal{T}_E$ denote the tube domain with the end vertex $\mathbf{v}_E$ and $\mathbf{v}_{E-}$. Let K denote the intersection of $\partial\mathcal{T}_E$ with the distanced compact Γ_E-invariant domain by Theorem 3.5 and K^* the closure of the boundary of the component of $\partial\mathcal{T}_E - K$ containing $\mathbf{v}_E$ in the boundary. Let C_1 be the convex hull of K^* in the tube domain $\mathcal{T}_E$.

It is standard to show that $\mathrm{bd}C_1 - K^*$ is a union of i-simplicies ($i \leq n$) in the interior $\mathcal{T}_E^o$ of the tube with vertices in K^*. Then each of the simplicies has the interior that is either disjoint from $\mathrm{bd}\tilde{\mathcal{O}}$ or is contained in it entirely: Let σ be a simplex in the boundary of K^*. Suppose that σ^o meets $\mathrm{bd}\tilde{\mathcal{O}}$ and is not contained in it entirely. We can find a segment $s \subset \sigma^o$ with a point z so that a component s_1 of $s - \{z\}$ is is in $\mathrm{bd}\tilde{\mathcal{O}}$ and the other component s_2 is disjoint from it. We may perturb s so that the new segment s' meets $\mathrm{bd}\tilde{\mathcal{O}}$ only in its interior. This contradicts the fact that $\tilde{\mathcal{O}}$ is convex.

Suppose that $\mathrm{bd}C_1 - K^*$ meets $\mathrm{bd}\tilde{\mathcal{O}}$. Then by the above discussions, there exists a line l in $\mathrm{bd}C_1 - K^*$ with endpoints x, y in K^* completely contained in $\mathrm{bd}\tilde{\mathcal{O}}$. There exists a triangle T with vertices $x, y, \mathbf{v}_E$ with $\partial T \subset \mathrm{bd}\tilde{\mathcal{O}}$. T^o is now a subset of an end neighborhood by assumption. Lemma 4.6 contradicts this. Therefore, we conclude that $\mathrm{bd}C_1 - K^*$ is contained in $\tilde{\mathcal{O}}$.

Let $z \in \mathcal{T}_{\mathbf{v}_E}^o$. By the uniform middle-eigenvalue condition and Proposition 4.4, the sequences of orbit elements of z limit to points of K^* only.

Now it is clear that $\mathrm{bd}C_1 - K^*$ has two components or C_1 is contained in a totally geodesic hypersurface with boundary K^*. Let us choose finitely many points $z_1, \dots, z_m \in \tilde{\mathcal{O}} - C_1$ in the two components. Then the convex hull $C_1' := C(\mathbf{\Gamma}_E(\{z_1, \dots, z_m\}))$. Proposition 4.4 shows that the orbits of z_i for each i accumulate to points of K^* only. Hence, there exists a totally geodesic hypersphere separating $\mathbf{v}_E$ with these orbit points

and one separating $\mathbf{v}_{E-}$ and the orbit points. Thus, C_1' is a compact convex set disjoint from $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ and $C_1' \cap \partial \mathcal{T}_E = K^*$.

We can choose sufficiently many points so that $A \cup B := \mathrm{bd} C_1' - \mathrm{bd} \mathcal{T}_{\mathbf{v}_E}$ is disjoint from C_1 (by pushing $\mathrm{bd} C_1 - K^* \subset \tilde{\mathcal{O}}$ away from C_1 choosing points from one of the fundamental domain.) Here, A and B are two components. Let A be the one at a distance from $\mathbf{v}_E$.

Since $A/\mathbf{\Gamma}_E$ and $B/\mathbf{\Gamma}_E$ are diffeomorphic to Σ_E, they are both compact suborbifolds transversal to the radial foliation. By Proposition 4.4, the boundary of A is in K^* and so is the boundary of B. Since they bound $C_1/\mathbf{\Gamma}_E$ in U a product type end neighborhood, it follows that $C_1/\mathbf{\Gamma}_E$ is compact.

Suppose that there exists an infinite line in the convex hull C_1' a line l in $A \cup B$ to a point of $\partial \mathcal{T}_{\mathbf{v}_E}$. Assume $l \subset A$. The line l project to a line l' in Ω_E.

Since $C_1'/\mathbf{\Gamma}_E$ is compact, we choose a compact fundamental domain F in C_1' and choose a sequence $\{x_i \in l\}_{i=1,2,\dots}$ converging to the endpoint of l' in $\mathrm{bd}\Omega_E$. We choose $\gamma_i \in \mathbf{\Gamma}_{\mathbf{v}_E}$ so that $\gamma_i(x_i) \in F$ where $\{\gamma_i(l')\}$ converges to a segment l'_∞ with both endpoints in $\mathrm{bd}\Omega_E$. Hence, $\{\gamma_i(l)\}$ converges to a segment l_∞ in A. We can assume that for the endpoint z of l in A, $\gamma_i(z)$ converges to the endpoint p_1. Proposition 4.4 implies that the endpoint p_1 of l_∞ is in K^* also. Let t be the endpoint of l not equal to z. Then $t \in K^*$ since t is in the boundary A with limit points in K_1 by Proposition 4.4. Thus, $\gamma_i(t)$ converges to a point of K^* and both end points of l_∞ is in K^* and hence $l_\infty \subset C_1$. Since $l \subset A$, it follows that $l_\infty \subset A$. As A is disjoint from C_1, this is a contradiction.

Since A and analogously B do not contain any geodesic of infinite length, $\mathrm{bd} C_1' - \mathrm{bd} \mathcal{T}_{\mathbf{v}_E}$ is a union of compact $n-1$-dimensional simplicies meeting one another in strictly convex dihedral angles. By choosing $\{z_1, \dots, z_m\}$ sufficiently close to $\mathrm{bd} C_1$, we may assume that $\mathrm{bd} C_1' - \mathrm{bd} \mathcal{T}_{\mathbf{v}_E}$ is in $\tilde{\mathcal{O}}$. Now by smoothing we obtain two boundary components of a lens. (Actually the condition can replace the definition of the lens condition.)

When $\mathbf{\Gamma}_E$ is reducible, we do the above arguments to show that each factor $\mathbf{\Gamma}_i$ acts on a compact convex set K_i' in $B(K_i)$ distanced from $\mathbf{v}_E$ and $\mathbf{v}_{E-}$. We obtain a join $K_1' * \cdots * K_{l_0}'$ where $\mathbf{\Gamma}_E$ acts on naturally.

Similar to the reasoning in the proof of Theorem 3.13, the end is totally geodesic: Let $z \in \mathcal{T}_{\mathbf{v}_E}^o$. (Basically, each element g in the center $\mathbb{Z}^{l_0 - 1}$ acts fixing each point of totally geodesic plane meeting $B(K_i)$ and K_i' has to be the intersection since $\lambda_1(g) > \lambda_{\mathbf{v}_E}(g)$ for g fixing the subspace corresponding to K_i. The join of $K_1', \dots, K_{l_0}'$ is totally geodesic compact convex set where $\mathbf{\Gamma}_E$ acts.)

Again, we need to do argument similar to above to find a lens and to complete the proof: that is, we find the convex hull C_1' as above and the top and the bottom hypersurface boundary components A and B. By the uniform middle-eigenvalue condition and Proposition 4.4, the orbits of z limit to points of K^* only.

(i) $\Rightarrow$ (ii): First, we show that the lens condition implies that $\mathbf{\Gamma}_E$ satisfies the middle-eigenvalue condition that $\lambda_1(g)/\lambda_{\mathbf{v}_E}(g) > 1$ for every g: There exists a lens-domain D where $D \cap s$ is a singleton for each maximal segment in $\mathrm{bd}\mathcal{T}_E$ with endpoints $\mathbf{v}_E$ and $\mathbf{v}_{E-}$. Let g be an element of $\mathbf{\Gamma}_E$. Take a maximal segment l in the lens domain D in a segment from $\mathbf{v}_E$ to $\mathbf{v}_{E-}$. Let p be the corresponding point in Ω_E. Then $g^m(p)$ converges

to a point p' of $\mathrm{bd}\Omega_E$. Then as $m \to \infty$, $\{g^m(l)\}$ converges to a point y corresponding to $l' \cap D$ the segment l' corresponding to p' since D is g-invariant. This implies $\lambda_1(g^m)/\lambda_{\mathbf{v}_E}(g^m) > 1$ for sufficiently large m as can be shown by simple computations. Hence, we obtain $\lambda_1(g)/\lambda_{\mathbf{v}_E}(g) > 1$ and the middle-eigenvalue condition.

If $\mathbf{\Gamma}_E$ satisfies the middle-eigenvalue condition, then so does its factors. Again assume that $\mathbf{\Gamma}_E$ is hyperbolic or is a center group in a bigger end fundamental group. Suppose that $\mathbf{\Gamma}_E$ does not satisfy the uniform middle-eigenvalue condition. Then we have $\tilde\Lambda_1^h(c) = \frac{n+1}{n}\mu^h(c)$ for a geodesic current c. By a small change of h so that μ^h becomes bigger near $c \in \mathcal{C}_1(E)$ fixing the linear part, we obtain that $\tilde\Lambda_1^h(g) < \frac{n+1}{n}\mu^h(g)$ for some $g \in \Gamma$. We know that a small perturbation of a lens-shaped end remains lens-shaped and in particular distanced by Theorem 3.15. However, we obtain that $\lambda_1(g) < \lambda_{\mathbf{v}_E}(g)$ for some g. The existence of the distanced lens domain and this implies that U is not properly convex as $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ are in the limits of $g^m(z)$ and $g^m(z')$ for some sets z, z'. Hence, we have $\tilde\Lambda_1^h(c) > 2\mu^h(c)$ for all currents c, $c \neq 0$. We obtain the uniform middle-eigenvalue condition.

This proves the uniform middle-eigenvalue conditions for each factor including the center factor if $\mathbf{\Gamma}_E$ is reducible. Hence, the uniform middle-eigenvalue condtion follows for $\mathbf{\Gamma}_E$.

□

Lemma 4.6. *Suppose that $\mathcal{O}$ is a properly convex real projective manifold with radial ends and satisfies the triangle condition. Every triangle T with $\partial T \subset \mathrm{bd}\tilde{\mathcal{O}}$ contained in an inverse image of an end neighborhood in an end neighborhood system has no vertex equal to an end vertex.*

Proof. Let $\mathbf{v}_E$ be an end vertex. Choose a fixed radially foliated end neighborhood system. Suppose that a triangle T with $\partial T \subset \mathrm{bd}\tilde{\mathcal{O}}$ contains a vertex equal to an end vertex. We take an end neighborhood system containing all triangles T' with $\partial T' \subset \mathrm{bd}\tilde{\mathcal{O}}$. Let U be an inverse image of an end neighborhood $\hat{U}$ in the end neighborhood system corresponding to E with an end vertex $\mathbf{v}_E$. The end orbifold Σ_E is a properly convex end of an orbiford $\mathcal{O}$.

Choose a maximal line l in T with endpoints $\mathbf{v}_E$ and w in the interior of an edge of T not containing $\mathbf{v}_E$. Then this line has to pass a point of the boundary of U and in T^o by definition of the radial foliations of the end neighborhoods. This implies that T^o is not a subset of an end neighborhood. This contradicts the assumption. □

We now generalize Propositon 4.2 to:

Corollary 4.7. *We are given a properly convex end E of a properly convex orbifold $\mathcal{O}$ with radial ends. Assume that $\mathcal{O}$ satisfies the triangle condtion for all of its properly convex real projective structures with radial ends. Then the subset of*

$$\mathrm{Hom}_{\mathbf{v}_E}(\pi_1(E), \mathrm{SL}_\pm(n+1, \mathbb{R}))$$

consisting of representations satisfying the uniform middle-eigenvalue condition is open.

Proof. This follows by Theorems 4.5 and 3.15.

□

Again, note that the assumption will be true for orbifolds with the relatively hyperbolic fundamental group (see [13]).

4.4. **Equivalence of the weak uniform middle-eigenvalue condition and the uniform eigenvalue conditions for properly convex ends.** The following proves Theorem 0.1 for properly convex ends by Theorem 4.5.

Proposition 4.8. *Let $\mathcal{O}$ be a properly convex real projective orbifold with radial ends. Assume that $\mathcal{O}$ satisfies the triangle condtion. Suppose that $\tilde{\mathcal{O}}$ is properly convex and satisfies the finite-essential-annulus condition or $\mathbf{\Gamma}$ is not virtually reducible. Let $\mathbf{\Gamma}_E$ the holonomy group of a properly convex end E of a properly convex real projective orbifold with radial ends. Then the following statements are equivalent :*

(i) $\mathbf{\Gamma}_E$ *satisfies the weak middle-eigenvalue condition.*
(ii) $\mathbf{\Gamma}_E$ *satisfies the uniform middle-eigenvalue condition.*

Proof. Clearly (ii) implies (i).

Now we prove (i) implies (ii): If $\mathbf{\Gamma}_E$ is irreducible, then $\mathbf{\Gamma}_E$ satisfies (ii) since $\mathbf{\Gamma}_E$ has trivial center virtually.

Suppose that $\mathbf{\Gamma}_E$ is reducible. Then we are in the situation similar to the proof of Theorem 3.13: Let $S_1, ..., S_{l_0}$ be the projective subspaces in general position meeting only at the end vertex $\mathbf{v}_E$ where factor groups $\mathbf{\Gamma}_1, ..., \mathbf{\Gamma}_{l_0}$ act irreducibly on. Let C_i' denote the union of segments of length π from $\mathbf{v}_E$ corresponding to the invariant cones in S_i where $\mathbf{\Gamma}_i$ acts irreducibly for each i. Let $g \in \mathbb{Z}^l$. Then

- $g|C_i'$ is either identity or
- $g|C_i'$ fixes a hyperspace $P_i \subset S_i$ not passing through $\mathbf{v}_E$ and g has a representation as a scalar multiplication in the affine subspace $S_i - P_i$ of S_i. Since g commutes with every element of $\mathbf{\Gamma}_i$ acting on C_i', it follows that $\mathbf{\Gamma}_i$ acts on P_i as well. We let $D_i' = C_i' \cap P_i$.

There exists a $\mathbf{\Gamma}_{E_i}$-invariant convex set K_i in C_i' distanced from $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$ by Theorem 3.5 provided $\mathbf{\Gamma}_{E_i}$ is a nontrivial hyperbolic group. By Lemma 4.9, the first case does not occur. For K_i in the second case, K_i has to be in a hypersurface and equals D_i'. Hence, there exists a minimal totally geodesic hypersurface P containing $D' := D_1' * \cdots * D_{l_0}'$. Moreover $D' \subset \mathrm{Cl}(U)$ since each D_i' is in $\mathrm{Cl}(U)$ by the existence of dense attracting fixed points of $\mathbf{\Gamma}_{E_i}$.

Let us take a fundamental domain F in D' of $\mathbf{\Gamma}_E$. We take the set F' finitely many points near F in both components of $\tilde{\mathcal{O}} - D'$. Then we consider the convex hull C' of $\bigcup_{g \in \mathbf{\Gamma}_E} g(F')$ The boundary $\partial C'$ of C' has one or two components, say A' and B'. At least the outer boundary component A' exists since $\tilde{\mathcal{O}}$ is a properly convex domain. We take sufficiently many points in F' so that A' is disjoint from P. Also, from $\mathbf{v}_E$, each segment connecting $\mathbf{v}_E$ with $\mathbf{v}_{E-}$ meets A' at exactly one-point by convexity. Hence, A' and Σ_E are diffeomorphic.

Since A' covers a closed orbifold diffeomorphic to Σ_E, it follows that there exists a fundamental domain F'' of A' with respect to $\mathbf{\Gamma}_E$. We claim that $\mathrm{bd} A' = \partial D'$: Suppose that we have a point $x' \in \mathrm{bd} A' - \partial D'$. Then x' lies on a maximal segment s from $\mathbf{v}_E$ in the direction of $\partial \Omega_E$ and hence in the direction of $\partial D'$. Let x'' be a point of

$s \cap \partial D'$. Then x'' is in a join D'' of a minimal number of D'_i that is a proper subset of $\{D'_1, \dots, D'_{l_0}\}$. We can choose a sequence $\{g_j \in \mathbb{Z}^{l_0-1}\}$ of distinct elements so that g_j has the largest eigenvalue at the points in D'' and $g_j|D'' \to \mathrm{I}$ since $\mathbb{Z}^{l_0-1}$ is diagonalizable with constant eigenvalues at each subspaces S_i and $\mathbb{R}^{l_0-1}/\mathbb{Z}^{l_0-1}$ is compact. Note that $\{g_j|D''\}$ is a bounded sequence and $\{\lambda_1(g_j)/\lambda_{\mathrm{v}_E}(g_j)\} \to \infty$ by Lemma 4.9. Then $g_j^{-1}(s)$ contains $g_j^{-1}(x'')$, the latter form a sequence converging to x'' in the interior of D'' as $g_j^{-1}|D''$ is bounded. Since we have

$$\frac{\lambda_{\mathrm{v}_E}(g_j^{-1})}{\lambda_1(g_j^{-1})} \to \infty,$$

$\{g_j^{-1}(s)\}$ converges to a complete ray passing x''. This contradicts the proper convexity of $\tilde{\mathcal{O}}$. Thus, $\mathrm{bd}A' \subset \partial D$. The converse $\partial D \subset \mathrm{bd}A'$ follows since A' meets every ray in the direction of D.

We claim that A' cannot have a geodesic of infinite length in the intrinsic metric $d_{\tilde{\mathcal{O}}}$ induced from one on $\mathcal{O}$. Suppose not. Let l be a geodesic in A'. If the both end points of l are infinite, then ∂l is in P and hence l is a subset of D'. This is absurd. So l is half-infinite. Let $l(t)$ parameterized t where $l(t)$ converges to its infinite end point as $t \to \infty$. Let us choose a sequence $t_i \to \infty$.

For each i, there exists a deck transformation g_i for each i so that $g_i(l(t_i)) \in F''$. Then since $g_i(l)$ passes F'', it follows that we can find a subsequence converging to a geodesic l' in A' that is infinite in both directions. This is a contradiction again.

Therefore, A' is a union of finite union T_i, $i \in I$, of compact totally geodesic $(n-1)$-simplices meeting each other in angles $< \pi$. One can then smooth this to obtain a strictly convex hypersurface isotopied from A'. Thus, we have a lens-type end. By Theorem 4.5, this implies that $\mathbf{\Gamma}_E$ satisfies the uniform middle-eigenvalue condition.

□

Lemma 4.9. *Suppose that $\mathcal{O}$ is a properly convex real projective orbifold with radial ends and satisfies the finite-essential-annulus condition or is not virtually reducible. Suppose that each end satisfies the weak uniform middle-eigenvalue condition. Then*

(i) *for an element g in the center $\mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$ we have $\lambda_1(g) > \lambda_{\mathrm{v}_E}(g)$ for the largest norm eigenvalue $\lambda_1(g)$ of g.*

(ii) *Also, for every sequence $\{g_j\}$ of distinct elements of $\mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$, we have*

$$\frac{\lambda_1(g_j)}{\lambda_{\mathrm{v}_E}(g_j)} \to \infty.$$

Proof. We continue to use notations of the proof of Proposition 4.8.

(i) Let $g \in \mathbb{Z}^l - \{\mathrm{I}\}$. Suppose that $\lambda_1(g) = \lambda_{\mathrm{v}_E}(g)$. Then g acts as an identity on the subspace C'_i corresponding to an irreducible factor K_i of K. Since g acts trivially on each D'_j for each j for all $j \neq i$ by (i). By Lemma 4.10 for the case of $l_0 = 2$, we obtain that $\mathrm{Cl}(\Omega)$ is a nontrivial join.

There exists a finite index subgroup $\mathbf{\Gamma}'$ of $\mathbf{\Gamma}$ so that it acts on each convex domain $L_1, \dots, L_m$ so that $\mathrm{Cl}(\Omega) = L_1 * \cdots * L_m$. Hence, if we assume that $\mathbf{\Gamma}$ is virtually irreducible, then this is absurd.

Now, we do not assume that $\mathbf{\Gamma}$ is virtual irreducible. Note that $g(\mathbf{v}_E)$ for $g \in \Gamma$ has to be in one of L_i since the corresponding end is properly convex and cannot have a nonproperly convex transversal structure for $g(\Omega_E)$. Since the number of end vertices of form $g(\mathbf{v}_E)$, $g \in \mathbf{\Gamma}$ are infinite and infinitely many of form $g(\mathbf{v}_E)$ are in L_i for some i. We may assume that $\mathbf{v}_E \in L_i$ by multiplying L_i by the inverse of one such element.

Choose one $g \in \mathbf{\Gamma}$ so that $g(\mathbf{v}_E)$ belong to same L_i as $\mathbf{v}_E$ and we have $\mathbf{v}_E \neq g(\mathbf{v}_E)$ since there are infinitely many such g. Let g_1 be a nontrivial element of the center of $\mathbf{\Gamma}_{\mathbf{v}_E}$ so that $\lambda_1(g_1) = \lambda_{\mathbf{v}_E}(g_1)$. Note that L_i is either in C'_j or D'_j for some j by construction using the notation in the proof of Proposition 4.8. Then g_1 fixes each point of L_i by construction of C'_j and D'_j. Note that $\mathrm{Cl}(U)$ is a manifold with boundary $(\mathrm{bd} U \cap \tilde{\mathcal{O}}) \cup (\mathrm{Cl}(U) \cap \mathrm{bd}\tilde{\mathcal{O}})$. This implies that $g_1(g(U)) \cap g(U) \neq \emptyset$ for an end neighborhood U of $\mathbf{v}_E$ since $g(\mathrm{Cl}(U)) \cup (g(\mathrm{Cl}(U)) \cap L_i)$ from a neighborhood of the interior of the g_1-invariant domain $g(\mathrm{Cl}(U)) \cap L_i$. Since $\mathbf{\Gamma}_{g(\mathbf{v}_E)} = g\mathbf{\Gamma}_{\mathbf{v}_E}g^{-1}$, it follows that $g_1 \in g\mathbf{\Gamma}_{\mathbf{v}_E}g^{-1}$ for infinitely many $g \in \mathbf{\Gamma}_E$. This contradicts our assumption.

(ii) Suppose that for a sequence g_j of $\mathbb{Z}^l - \{\mathrm{I}\}$, we have $\{\lambda_1(g_j)/\lambda_{\mathbf{v}_E}(g_j)\}$ is bounded. Since $\lambda_1(g_j) \geq \lambda_{\mathbf{v}_E}(g_j)$, we assume without loss of generality that $\lambda_1(g_j)$ occur for a fixed C'_i by taking a subsequence of $\{g_j\}$ if necessary. Then $\{g_j\}$ acts as a bounded set of projective automorphisms of C'_i. Since g_j acts trivially on each D'_j for each j for all $j \neq s$ by (i). Again by Lemma 4.10 for the case of $l_0 = 2$, we see that $\mathrm{Cl}(\Omega)$ is a nontrivial join and this leads to contradiction as above.

□

Lemma 4.10. *Suppose that a set G of projective automorphisms acts on subspaces $S_1, \dots, S_{l_0}$ and a properly convex domain Ω corresponding to subspaces $V_1, \dots, V_{l_0}$ so that $V_i \cap V_j = \{0\}$ for $i \neq j$ and $V_1 \oplus \cdots \oplus V_{l_0} = \mathbb{R}^{n+1}$. We assume that for each S_i, $G_i := \{g|S_i | g \in G\}$ form a bounded set of automorphisms, and that for each S_i, there exists a subsequence $\{g_{i,j} \in G\}$ with largest norm eigenvalue $\lambda_{i,j}$ restricted at G_i so that we have $\{\lambda_{i,j}\} \to \infty$. Let $\Omega_i := \mathrm{Cl}(\Omega) \cap S_i$. Then we obtain $\mathrm{Cl}(\Omega) = \Omega_1 * \cdots * \Omega_{l_0}$ for $\Omega_j \neq \emptyset, j = 1, \dots, l_0$.*

Proof. Let $z = [\vec{v}_z]$ for a vector in $\mathbb{R}^n$. We write $\vec{v}_z = \vec{v}_1 + \cdots + \vec{v}_{l_0}$, $\vec{v}_j \in V_j$ for each j $j = 1, \dots, l_0$, which is a unique sum. Then $z_i = [v_i]$ is uniquely determined by z.

Let $z \in \Omega$. We choose a subsequence of $\{g_{i,j}\}$ so that $\{g_{i,j}|S_i\}$ converges to a projective automorphism $g_{i,\infty} : S_i \to S_i$ and $\lambda_{i,j} \to \infty$ as $j \to \infty$. Then $g_{i,\infty}$ also acts on Ω_i. Then $g_{i,j}(z_i) \to g_{i,\infty}(z_i) = z_{i,\infty}$ for a point $z_{i,\infty} \in S_i$. We have $g_{i,j}(z) \to z_{i,\infty}$ by the eigenvalue condition. Thus, we obtain $z_{i,\infty} \in \Omega_i = \mathrm{Cl}(\Omega) \cap S_i$ as $z_{i,\infty}$ is the limit of a sequence of orbit points of z. Hence we also obtain $z_i \in \Omega_i$ and $\Omega_i \neq \emptyset$ as

$$z_i = g_{i,\infty}^{-1}(g_{i,\infty}(z_i)) = g_{i,\infty}^{-1}(\lim_j g_{i,j}(z_i)) = g_{i,\infty}^{-1}(z_{i,\infty}) \in \Omega_i. \tag{11}$$

We prove that Ω is a subset of the join. Let $z \in \Omega$. For each i by the eigenvalue conditions, we choose a sequence $\{g_{i,j}\}, j = 1, 2, \dots$ so that $g_{i,j}(z) \to z_{i,\infty} \in \Omega_i$ for each $i = 1, \dots, l_0$ by the invariance of Ω. Since $z_1, \dots, z_{l_0}$ is in $\Omega_1, \dots, \Omega_{l_0}$ respectively, it follows that $z \in \Omega_1 * \cdots * \Omega_{l_0}$. Thus, we obtain $\mathrm{Cl}(\Omega) \subset \Omega_1 * \cdots * \Omega_{l_0}$.

Conversely, for each point z_i of Ω_i, one can construct z as above. Thus, $\Omega_i \subset \mathrm{Cl}(\Omega)$ by equation 11. We obtain that $\Omega_1 * \cdots * \Omega_{l_0}$ is a subset of $\mathrm{Cl}(\Omega)$.

□

5. Non-properly convex ends

We will now study the ends where the transverse real projective structures are not properly convex. Let E be an end of $\mathcal{O}$ and let U the corresponding end neighborhood in $\tilde{\mathcal{O}}$ with the end vertex $\mathbf{v}_E$. Let Ω_E denote the universal cover of the end orbifold Σ_E as a domain in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$.

Suppose that Ω_E is not properly convex. Then the closure $\mathrm{Cl}(\Omega_E)$ contains a great (i_0-1)-dimensional sphere and Ω_E is foliated by i_0-dimensional hemispheres with this boundary. Let $\mathbb{S}^{i_0-1}_\infty$ denote the great (i_0-1)-dimensional sphere in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ of Ω_E. The space of i_0-dimensional hemispheres in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ with boundary $\mathbb{S}^{i_0-1}_\infty$ form a projective sphere $\mathbb{S}^{n-i_0-1}$. The projection

$$\Pi_K : \mathbb{S}^{n-1}_{\mathbf{v}_E} - \mathbb{S}^{i_0-1}_\infty \to \mathbb{S}^{n-i_0-1}$$

gives us an image K that is a properly convex set. (See [10] for details.)

Let $\mathbb{S}^{i_0}_\infty$ be a great i_0-dimensional sphere containing $\mathbf{v}_E$ corresponding to the directions of $\mathbb{S}^{i_0-1}_\infty$ from $\mathbf{v}_E$. The space of (i_0+1)-dimensional hemispheres with boundary $\mathbb{S}^{i_0}_\infty$ again has the structure of the projective sphere $\mathbb{S}^{n-i_0-1}$, identifiable with the above one. Denote by $\mathbf{Aut}_{\mathbb{S}^{i_0}_\infty}(\mathbb{S}^n)$ the group of projective automorphisms of $\mathbb{S}^n$ acting on $\mathbb{S}^{i_0}_\infty$ and fixing $\mathbf{v}_E$.

Each i_0-dimensional hemisphere H^{i_0} in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ with $\partial H^{i_0} = \mathbb{S}^{i_0-1}_\infty$ corresponds to an (i_0+1)-dimensional hemisphere H^{i_0+1} in $\mathbb{S}^n$ with common boundary $\mathbb{S}^{i_0}_\infty$ that contains $\mathbf{v}_E$. Suppose that $\mathbb{S}^{i_0}_\infty$ is $h(\pi_1(E))$-invariant. We let N be the subgroup of $h(\pi_1(E))$ of elements inducing trivial actions on $\mathbb{S}^{n-i_0-1}$. The above exact sequence

$$1 \to N \to h(\pi_1(E)) \xrightarrow{\pi_K} N_K \to 1$$

is so that kernel normal subgroup N acts trivially on $\mathbb{S}^{n-i_0-1}$ but acts on each hemisphere with boundary equal to $\mathbb{S}^{i_0}_\infty$ and N_K acts faithfully by the action induced from π_K.

Here N_K is a subgroup $\mathbf{Aut}(K)$ of the group of projective automorphisms of K.

Theorem 5.1. *Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a topologically tame properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\hat{h}_E : \pi_1(E) \to \mathbf{Aut}(\mathbb{S}^{n-1}_{\mathbf{v}_E})$ be the associated holonomy homomorphism induced by restriction h_E of the holonomy homorphisms $h : \pi_1(\mathcal{O}) \to \mathbf{Aut}(\mathbb{S}^n)$ to $\pi_1(E)$ and considering the action on $\mathbf{Aut}(\mathbb{S}^{n-1}_{\mathbf{v}_E})$ for the corresponding end vertex $\mathbf{v}_E$. Then*

- *Σ_E is foliated by complete affine subspaces of dimension i_0, $i_0 > 0$.*
- *$h(\pi_1(E))$ fixes the great sphere $\mathbb{S}^{i_0-1}_\infty$ of dimension i_0-1 in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$.*
- *There exists an exact sequence*

$$1 \to N \to \pi_1(E) \xrightarrow{\pi_K} N_K \to 1$$

where N acts trivially on quotient great sphere $\mathbb{S}^{n-i_0-1}$ and N_K acts faithfully on a properly convex domain K in $\mathbb{S}^{n-i_0-1}$ isometrically with respect to the Hilbert metric d_K.

Proof. These are proved in Section 1.4 of [10]. □

We denote by $\mathcal{F}$ the foliations on Σ_E or the corresponding one in Ω_E.

We denote by $\mathbf{\Gamma}_E$ the end fundamental group acting on U fixing $\mathbf{v}_E$. Denote the induced foliations on Σ_E and Ω_E by $\mathcal{F}_E$. For each element $g \in \mathbf{\Gamma}_E$, we define $\mathsf{length}_K(g)$ to be $\inf\{d_K(x, g(x))|x \in K\}$.

Then for some $l' \geq 1$, $\mathbf{\Gamma}_E$ is isomorphic to $\mathbb{Z}^{l'-1} \times \mathbf{\Gamma}_1 \times \cdots \times \mathbf{\Gamma}_{l'}$ up to finite index where each $\mathbf{\Gamma}_i$ acts irreducibly on K_i for each $i = 1, \dots, r$. We assume that $\mathbf{\Gamma}_i$ is hyperbolic for $i = 1, \dots, s$ and $\mathbf{\Gamma}_i = \{\mathrm{I}\}$ for $s + 1 \leq i \leq l'$.

Assume that K has a decomposition into $K_1 * \cdots * K_{l_0}$ for properly convex domains K_i, $i = 1, \dots, l_0$. Let $K_i, i = 1, \dots, s$, be the ones with dimension ≥ 2. N_K is virtually isomorphic to the product

$$\mathbb{Z}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_s$$

where Γ_i is obtained from N_K by restricting to K_i and A is a free abelian group of finite rank. We will divide into two cases later where N_K is discrete and indiscrete. In the beginning of Section 5.1, we show that N is virtually nilpotent when N_K is discrete. In Section 5.6, we show that N is virtually nilpotent when N_K is indiscrete. Thus, $\mathbf{\Gamma}_i \cap N$ is virtually nilpotent and a normal subgroup of $\mathbf{\Gamma}_i$. If $\mathbf{\Gamma}_i$ is hyperbolic, then this group has to be trivial. Therefore, we obtain that each $\mathbf{\Gamma}_i$ for $i = 1, \dots, s$, is mapped isomorphic to some Γ_i in N_K provided $\mathbf{\Gamma}_i$ is hyperbolic. Thus, each K_i is a strictly convex domain or a point by the results in [1].

We also assume that the *uniform middle-eigenvalue condition* relative to N that for each element $\pi_K(g) \in \Gamma_i - \{\mathrm{I}\}$ for each factor Γ_i of N_K, we have the eigenvalue $\lambda_{\mathbf{v}_E}(g)$ at $\mathbf{v}_E$ and the largest norm eigenvalue $\lambda_1(g)$ at points other than $\mathbf{v}_E$ where

- there exists a constant $C > 0$ independent of g and i such that for $\pi_K(g) \in \Gamma_i - \{\mathrm{I}\}$ or $\pi_K(g)$ in the central set $\mathbb{Z}^{l_0-1} - \{\mathrm{I}\}$

$$C^{-1}\mathsf{length}_K(g) \leq \log \frac{\lambda_1(g)}{\lambda_{\mathbf{v}_E}(g)} \leq C\mathsf{length}_K(g). \tag{12}$$

For elements of the center $\mathbb{Z}^{l_0-1}$, we may just require $\lambda_1(g) \geq \lambda_{\mathbf{v}_E}(g)$ for all $g \in \mathbb{Z}^{l_0-1}$. In this case, we say that Γ satisfies the *weakly uniform middle-eigenvalue condition*.

This condition implies that each irreducible factor of N_K satisfies the uniform middle-eigenvalue condition provided it acts on $B(K_i) \subset B(K)$ where $B(K_i)$ is the tube domain corresponding to K_i. We will show that N_K acts on $B(K)$ later under this assumption. It is a generalization of the uniform middle-eigenvalue condition. (Also, this definition is made so that the limits of end holonomy representations can be understood.)

Definition 5.2. Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\mathbf{\Gamma}_E$ be the end fundamental group. Let $\lambda_0(g)$ denote the largest norm of the eigenvalue of $g \in \Gamma_E$ not associated with a vector in the direction of $\mathbb{S}^{i_0}_\infty$. Also, let $\lambda_{n+1}(g)$ denote the smallest one not associated with any vector in direction of $\mathbb{S}^{i_0}_\infty$. Let $\lambda(g)$ be the largest of the norm of the eigenvalue of g occurring in $\mathbb{S}^{i_0}_\infty$ and $\lambda'(g)$ the smallest one.

Any element of g has a Jordan decomposition. An irreducible Jordan-block corresponds to a unique subspace in $\mathbb{C}^{n+1}$, called a Jordan subspace. We define the *real sum* of Jordan-block subspaces is defined to be the intersection with $\mathbb{R}^{n+1}$ the sum of the complex subspaces invariant under the conjugation.

We will sharpen the following to inequality in the discrete and indiscrete cases.

Proposition 5.3. *Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\mathbf{\Gamma}_E$ be the end fundamental group. Let $g \in \mathbf{\Gamma}_E$. Then we have*

$$\lambda_0(g) \geq \lambda(g) \geq \lambda'(g) \geq \lambda_{n+1}(g),$$

and an infinite order element of $\mathbf{\Gamma}_E$ always has a fixed point in $\mathrm{bd} K$ for largest norm eigenvalue and a fixed point in $\mathrm{bd} K$ of smallest norm eigenvalue.

Proof. We may assume that g is of infinite order. Suppose that $\lambda(g) > \lambda_0(g)$. Then we have $\lambda(g) \geq \lambda_{\mathbf{v}_E}(g)$ where $\lambda(g)$ is the largest norm of the eigenvalues.

If we have $\lambda(g) = \lambda_{\mathbf{v}_E}(g)$, then $\lambda_{\mathbf{v}_E}(g) > \lambda_0(g)$ contradicts the middle-eigenvalue condition. Thus, $\lambda(g) > \lambda_{\mathbf{v}_E}(g)$. Hence, there exists a great sphere $\mathbb{S}^j$ in $\mathbb{S}^{i_0}_\infty$ distinct from $\mathbf{v}_E$ or $\mathbf{v}_{E-}$ corresponding to the real sum of Jordan-block subspace of g corresponding to norm $\lambda(g)$ where j could be zero. Since $\mathbf{v}_E \in \mathbb{S}^{i_0}_\infty$ is not contained in $\mathbb{S}^j$, we obtain $j + 1 \leq i_0$.

There exists a g-invariant great sphere C_1 containing $\mathbf{v}_E$ disjoint from $\mathbb{S}^j$ corresponding to the real sum of Jordan-block subspaces where g has strictly smaller norm eigenvalues. Then C_1 is of dimension $n - 1 - j$ and contains $\mathbf{v}_E$.

Considering the sphere $\mathbb{S}^{n-1}_{\mathbf{v}_E}$ at $\mathbf{v}_E$, we have C_1 goes to a subspace C_1' of dimension $n - j - 2$ in $\mathbb{S}^{n-1}_{\mathbf{v}_E}$. Each complete affine leaf l of Ω_E has dimension i_0 where $i_0 \geq j + 1$. Since

$$\dim l + \dim C_1' = i_0 + n - j - 2 \geq n - 1 = \dim \Omega_E,$$

and a vector space consideration shows that each leaf meets C_1'. Thus, we have now $C_1 \cap U \neq \emptyset$.

Let us define $U_1 := C_1 \cap U$, a nonempty convex set open in C_1. Take a small ball B' meeting U_1. Then $\{g^k(B')\}$ converges to a $(j+1)$-dimensional hemisphere with boundary $\mathbb{S}^j$: Since we can write any vector as a sum of vectors in the directions in C_1 or $\mathbb{S}^j$. The action of g^k as $k \to \infty$ makes the former vectors very small compared to the latter ones.

This contradicts the proper convexity of U as $g^k(B'') \subset U$ and the geometric limit is in $\mathrm{Cl}(U)$.

Also the consideration of g^{-1} completes the inequality.

The last part follows since $\lambda_0(g) > \lambda_{n+1}(g)$ for an infinite order element. The existence of fixed points in $\mathrm{bd} K$ follows since the action of g on K is semi-simple. □

5.1. **Discrete case.** Suppose that N_K is discrete subgroup of $\mathbf{Aut}(\mathbb{S}^{n-i_0-1})$. This is a much simpler case. (Actually N_K virtually equals the group

$$\mathbb{Z}^{l_0-1} \times \Gamma_1 \times \cdots \times \Gamma_{l_0}$$

since each factor Γ_i commutes with the other factors and act trivially on K_j for $j \neq i$ as was shown in the proof of Theorem 3.13 and N_K acts cocompactly on K.)

Since N act on each leaf l of $\mathcal{F}_E$ in Ω_E, it also acts on a properly convex domain $\tilde{\mathcal{O}}$ and $\mathbf{v}_E$ in a subspace $\mathbb{S}^{i_0+1}_l$ in $\mathbb{S}^n$ corresponding to l. $l/N \times \mathbb{R}$ is an open real projective orbifold diffeomorphic to $(\mathbb{S}^{i_0+1}_l \cap \tilde{\mathcal{O}})/N$. As above, Proposition 3.8 and Theorem 3.9 show that l corresponds to a horospherical end of $(\mathbb{S}^{i_0+1}_l \cap \tilde{\mathcal{O}})/N$ and N is virtually

nilpotent and N is a virtually a cocompact subgroup of a unipotent group, a conjugate into $\mathsf{SO}(i_0+1,1)$ and acting on an ellipsoid of dimension i.

By Malcev, the Zariski closure $Z(N)$ of N is a virtually unipotent Lie group with finitely many components and $Z(N)/N$ is compact. Let $\mathcal{N}$ denote the identity component of the Zariski closure of N so that $\mathcal{N}/(\mathcal{N}\cap N)$ is compact. N' acts on the great sphere $\mathbb{S}^{i_0+1}_l$ containing $\mathbf{v}_E$ and corresponding to l. Since N acts on a horoball in $\mathbb{S}^{i_0+1}_l$ and $\mathcal{N}/\mathcal{N}\cap N$ is compact, we can modify the horoball to be invariant under $\mathcal{N}$ by taking the convex hull of images of it under $\mathcal{N}$. We choose an arbitrary point $w \in \mathbb{S}^{i_0+1}_l \cap U$.

Let V^{i_0+1} denote the subspace corresponding to $\mathbb{S}^{i_0}_\infty$ containing $\mathbf{v}_E$ and V^{i_0+2} the subspace corresponding to $\mathbb{S}^{i_0+1}_l$. We can write each element $k\in\mathcal{N}$ as an $(n+1)\times(n+1)$-matrix

$$\left(\begin{array}{ccc} \mathrm{I}_{n-i_0-1} & 0 & 0 \\ \vec{0} & 1 & 0 \\ C_k & * & U_k \end{array}\right) \tag{13}$$

where $C_k > 0$ is a $(i_0+1)\times(n-i_0-1)$-matrix, U_k is a unipotent $(i_0+1)\times(i_0+1)$-matrix, 0 indicates various zero row or column vectors, $\vec{0}$ denote the zero row-vector of dimension $n-i_0-1$ and I_{n-i_0-1} is the $(n-i_0-1)\times(n-i_0-1)$ identity-matrix. This follows since element k acts trivially on $\mathbb{R}^{n+1}/V^{i_0+1}$ and k acts as a unipotent matrix on the subspace V^{i_0+2}. (We let w be a basis element corresponding to the $(n-i_0-1)$th column and $\mathbf{v}_E$ corresponds to the last column of form $(0,\dots,0,1)$.)

Since the argument will be carried out for a virtual subgroup of $\pi_1(E)$ and $\mathcal{N}$ is virtually abelian, we assume that $\mathcal{N}$ is abelian from now on.

Since $\mathcal{N}$ is abelian and acts on an ellipsoid with a complete Euclidean metric of dimension i, each element of $\mathcal{N}$ is a translation in some affine coordinate of Ω_E. Considering the great 2-sphere containing the translation subspace and $\mathbf{v}_E$ as a coordinate subspace, we see that an element of $\mathcal{N}$ can be written on a coordinate system of $\mathbb{S}^{i_0+1}$ as

$$\left(\begin{array}{cccc} 1 & 0 & \vec{0} & 0 \\ 1 & 1 & \vec{0} & 0 \\ \vec{0}^T & \vec{0}^T & \mathrm{I}_{i_0-1} & \vec{0}^T \\ \frac{1}{2} & 1 & \vec{0} & 1 \end{array}\right). \tag{14}$$

We can make each translation direction of generators of $\mathcal{N}$ in Ω_E to be one of the standard vector. Therefore, we can find a coordinate system of V^{i_0+2} so that the generators are of $(i_0+2)\times(i_0+2)$-matrix forms

$$\hat{\mathcal{N}}_j := \left(\begin{array}{ccc} 1 & \vec{0} & 0 \\ \vec{e}_j^T & \mathrm{I}_{i_0} & 0 \\ \frac{1}{2} & \vec{e}_j & 1 \end{array}\right) \tag{15}$$

where $\vec{e}_{j,k} = \delta_{jk}$ a row i-vector for $j = 1, \dots, i$. Hence, the generator $\mathcal{N}_j$ of $\mathcal{N}$ is of form

$$\mathcal{N}_j := \begin{pmatrix} \mathrm{I}_{n-i_0-1} & 0 & 0 \\ \vec{0} & 1 & 0 \\ C_j & v_j & N'_j \end{pmatrix} \tag{16}$$

where we used coordinates so that N'_j is a lower-triangular form in a $(i_0+1)\times(i_0+1)$-matrix and v_j is a column vector of dimension i_0+1 and is not all zero and C_j is a $(n-i_0-1)\times(i_0+1)$-matrix. (We can choose any $w \in \mathbb{S}^{i_0+1}_l \cap U$.)

We remark that $N \cap \mathcal{N} := \mathcal{N}(L)$ for a lattice L in $\mathbb{R}^{i_0}$.

Let g be an element of Γ mapping to a nontrivial element of N_K also acting on $\mathbb{S}^{i_0}_\infty$. Let $g' = g|\mathbb{S}^{i_0}_\infty$ and let U_g denote the corresponding matrix for V^{i_0+1}. Then $U_g N' U_g^{-1}$ is still an element of N' restricted to V^{i_0+1}. Thus, U_g belongs to a normalizer of the restriction of N' in V^{i_0+1}. The matrix of g can be written as

$$\begin{pmatrix} S_g & 0 \\ C_g & U_g \end{pmatrix} \tag{17}$$

where U_g is $(i_0+1)\times(i_0+1)$ normalizing matrix and S_g is an $(n-i_0)\times(n-i_0)$ semisimple matrix and C_g is $(n-i_0)\times(i_0+1)$-matrix. We call the subgroup

$$\left\{ \frac{1}{|\det(S_g)|^{\frac{1}{n-i_0}}} S_g | g \in h(\pi_1(E)) \right\} \subset \mathrm{SL}_\pm(n-i_0, \mathbb{R})$$

the semisimple part of $h(\pi_1(E))$ since it acts on a compact convex subset K discretely and properly discontinuously with compact quotient and has to be semisimple by the main results of [1].

Example 5.4. Let us consider two ends E_1, a lens type one, with the end neighborhood U_1 in the universal cover of a real projective orbifold $\mathcal{O}_1$ of dimension $n - i_0$ with the end vertex $\mathbf{v}_1$, and and E_2 the end neighborhood U_2 , a horospherical type one, in the universal cover of a real projective orbifold $\mathcal{O}_2$ of dimension i_0 with the end vertex $\mathbf{v}_2$. Let $\mathbf{\Gamma}_1$ denote the projective automorphism group in $\mathbf{Aut}(\mathbb{S}^{n-i_0})$ acting on U_1 corresponding to E_1. Let $\mathbf{\Gamma}_2$ denote the one in $\mathbf{Aut}(\mathbb{S}^{i_0})$ acting on U_2 unipotently and hence it is a cusp action. We assume that $\mathbf{\Gamma}_1$ fixes a great sphere $\mathbb{S}^{n-i_0-1} \subset \mathbb{S}^{n-i_0}$ disjoint from $\mathbf{v}_1$. There exists a properly convex open domain K where $\mathbf{\Gamma}_1$ acts cocompactly. We change U_1 to be the interior of the join of K and $\mathbf{v}_1$. We have $\mathbb{S}^{n-i_0}$ and $\mathbb{S}^{i_0}$ embedded in $\mathbb{S}^n$ meeting at $\mathbf{v} = \mathbf{v}_1 = \mathbf{v}_2$. We embed U_2 in $\mathbb{S}^{i_0}$ and $\mathbf{\Gamma}_2$ in $\mathbf{Aut}(\mathbb{S}^n)$ fixing each point of $\mathbb{S}^{n-i_0-1}$ so that $\mathbf{\Gamma}_2$ acts trivially on $\mathbb{S}^{n-i_0-1}$ and unipotently on $\mathbb{S}^{i_0}$. We can embed U_1 in $\mathbb{S}^{n-1}$ and $\mathbf{\Gamma}_1$ in $\mathbf{Aut}(\mathbb{S}^n)$ acting on the embedded U_1 so that $\mathbf{\Gamma}_1$ acts on $\mathbb{S}^{i_0}$ normalizing $\mathbf{\Gamma}_2$ and acting on U_1.

At least one can find some such embeddings by finding an arbitrary homomorphism $\rho : \mathbf{\Gamma}_1 \to N(\mathbf{\Gamma}_2)$ for a normalizer $N(\mathbf{\Gamma}_2)$ of $\mathbf{\Gamma}_2$ in $\mathbf{Aut}(\mathbb{S}^n)$. This construction does depend on ρ.

We find an element $\zeta \in \mathbf{Aut}(\mathbb{S}^n)$ fixing each point of the disjoint spheres $\mathbb{S}^{n-i_0-1}$ and acting on $\mathbb{S}^{i_0}$ as an unipotent element normalizing $\mathbf{\Gamma}_2$ so that the corresponding matrix has only two norms of eigenvalues. Then ζ centralizes $\mathbf{\Gamma}_1$ and normalizes $\mathbf{\Gamma}_2$. Let U be the join of U_1 and U_2, a properly convex domain. $U/\langle \mathbf{\Gamma}_1, \mathbf{\Gamma}_2, \zeta\rangle$ gives us an end of

dimension n diffeomorphic to $\Sigma_{E_1} \times \Sigma_{E_2} \times \mathbb{S}^1 \times \mathbb{R}$ and the transversal real projective manifold is diffeomorphic to $\Sigma_{E_1} \times \Sigma_{E_2} \times \mathbb{S}^1$. We call the results the *joined* end and the joined end neighborhoods.

For above, we required that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are discrete with compact stabilizers. Suppose that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are Lie groups and they have compact stabilizers, and we have a parameter of ζ^t for $t \in \mathbb{R}$ centralizing $\mathbf{\Gamma}_1$ and normalizing $\mathbf{\Gamma}_2$ and restricting a unipotent action on $\mathbb{S}^{i_0}$ acting on U_2. We obtain a *joined homogeneous action* of the semisimple and cusp actions. Let U be the properly convex open subset obtained as above as a join of U_1 and U_2. Let G denote the constructed Lie group by taking the embeddings of $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ as above. Given a discrete cocompact subgroup of G, we obtained a *joined end* by taking the quotient of U.

Example 5.5. Let N be as in equation 13. In fact, we let $C_1 = 0$ to simplify arguments and let N be a translational unipotent group in conjugate to $\mathsf{SO}(i_0+1, 1)$ corresponding to an i_0-dimensional Euclidean space.

We find a closed convex real projective orbifold Σ of dimension $n - i_0 - 1$ and find a homomorphism from $\pi_1(\Sigma)$ to a subgroup of $\mathbf{Aut}(\mathbb{S}^{i_0})$ normalizing N or even N itself. (We will use a trivial one to begin with.) Using this, we obtain a group Γ so that

$$1 \to N \to \Gamma \to \pi_1(\Sigma) \to 1.$$

Actually, we assume that this is split, i.e., $\pi_1(\Sigma)$ acts trivially on N.

We now consider an example where $i_0 = 1$. Let N be 1-dimensional and be generated by N_1 as in Equation 16.

$$N_1 := \left(\begin{array}{cc|cc} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \vec{0} & 1 & 0 & 0 \\ \hline \vec{0} & 1 & 1 & 0 \\ \vec{0} & \frac{1}{2} & 1 & 1 \end{array}\right) \tag{18}$$

where $i_0 = 1$ and we set $C_1 = 0$ for illustration here.

We take a discrete faithful proximal representation $\tilde{h} : \pi_1(\Sigma) \to \mathsf{GL}(n - i_0, \mathbb{R})$ acting on a convex cone C_Σ in $\mathbb{R}^{n-i_0}$. We define $h : \pi_1(\Sigma) \to \mathsf{GL}(n+1, \mathbb{R})$ by matrices

$$h(g) := \begin{pmatrix} \tilde{h}(g) & 0 & 0 \\ \vec{d}_1(g) & a_1(g) & 0 \\ \vec{d}_2(g) & c(g) & \lambda_{\mathbf{v}_E}(g) \end{pmatrix} \tag{19}$$

where $\vec{d}_1(g)$ and $\vec{d}_2(g)$ are $n - i_0$-vectors and $g \mapsto \lambda_{\mathbf{v}_E}(g)$ is a homomorphism as defined above for the end vertex and $\det \tilde{h}(g) a_1(g) \lambda_{\mathbf{v}_E}(g) = 1$. These are the most general form

possible:

$$h(g^{-1}) := \begin{pmatrix} \tilde{h}(g)^{-1} & \begin{matrix} 0 & 0 \end{matrix} \\ -\begin{pmatrix} \frac{\vec{d}_1(g)}{a_1(g)} \\ \frac{-c(g)\vec{d}_1(g)}{a_1(g)\lambda_{\mathbf{v}_E}(g)} + \frac{\vec{d}_2(g)}{\lambda_{\mathbf{v}_E}(g)} \end{pmatrix} \tilde{h}(g)^{-1} & \begin{matrix} \frac{1}{a_1(g)} & 0 \\ \frac{-c(g)}{a_1(g)\lambda_{\mathbf{v}_E}(g)} & \frac{1}{\lambda_{\mathbf{v}_E}(g)} \end{matrix} \end{pmatrix}. \tag{20}$$

Then the conjugation of N_1 by $h(g)$ gives us

$$\begin{pmatrix} \mathrm{I}_{n-i_0} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{pmatrix} \vec{0} & a_1(g) \\ \vec{*} & * \end{pmatrix} \tilde{h}(g)^{-1} & \begin{matrix} 1 & 0 \\ \frac{\lambda_{\mathbf{v}_E}(g)}{a_1(g)} & 1 \end{matrix} \end{pmatrix}. \tag{21}$$

Our condition on the form of N_1 shows that $(0, 0, \dots, 0, 1)$ has to be a common eigenvector by $\tilde{h}(\pi_1(E))$ and we also assume that $a_1(g) = \lambda_{\mathbf{v}_E}(g)$ for the reasons to be justified later and the last row of $\tilde{h}(g)$ equals $(\vec{0}, \lambda_{\mathbf{v}_E}(g))$. Thus, the semisimple part of $h(\pi_1(E))$ is reducible.

Some further computations show that we must and could take any $h : \pi_1(E) \to \mathrm{SL}(n - i_0, \mathbb{R})$ with matrices of form

$$h(g) := \left(\begin{array}{cc|cc} S_{n-i_0-1}(g) & 0 & 0 & 0 \\ \vec{0} & \lambda_{\mathbf{v}_E}(g) & 0 & 0 \\ \hline \vec{0} & c(g) & \lambda_{\mathbf{v}_E}(g) & 0 \\ \vec{0} & b(g) & c(g) & \lambda_{\mathbf{v}_E}(g) \end{array}\right). \tag{22}$$

for $g \in \pi_1(E) - N$ by a choice of coordinates by the semisimple property of the $(n - i_0) \times (n - i_0)$-upper left part of $h(g)$.

Since $\tilde{h}(\pi_1(E))$ has a common eigenvector, Benoist [2] shows that K in this case is decomposable and $N_K = N_K' \times \mathbb{Z}$ for another subgroup N_1' and the image of the homomorphism $g \in N_K' \to S_{n-i_0-1}(g)$ can be assumed to give a discrete projective automorphism group acting properly discontinuously on a properly convex subset K' in $\mathbb{S}^{n-i_0-2}$ with a compact quotient. Furthermore the generator of $\mathbb{Z}$ is central.

Let $\mathcal{E}$ be the one-dimensional ellipsoid where lower right 3×3-matrix of N_K acts on. From this, we easily see that the end is of the join form $K'/N_K' \times \mathbb{S}^1 \times \mathcal{E}/\mathbb{Z}$ by taking a double cover if necessary and $\pi_1(E)$ is isomorphic to $N_K' \times \mathbb{Z} \times \mathbb{Z}$ up to taking an index two subgroups. (In this case, N_K centralizes $\mathbb{Z} \subset N_K'$ and the second $\mathbb{Z}$ is in the centralizer of Γ.)

We can think of this as the join of K'/N_K' with $\mathcal{E}/\mathbb{Z}$ as K' and $\mathcal{E}$ are on disjoint complementary projective spaces of respective dimensions $n-3$ and 2 to be denoted $S(K')$ and $S(\mathcal{E})$ respectively. Obviously, we can sometimes generalize this example by taking higher dimension for i and letting C_g and U_g be more general.

5.1.1. *The standard quadric in $\mathbb{R}^i$ and the group acting on it.* Let us consider an affine subspace A^{i_0+1} of $\mathbb{S}^{i_0+1}$ with coordinates $x_0, x_1, \dots, x_{i_0+1}$ given by $x_0 > 0$. The standard quadric in A^{i_0+1} is given by

$$x_{i_0+1} = x_1^2 + \cdots + x_{i_0}^2.$$

Clearly the orthogonal maps $O(i_0)$ acting on the planes given by $x_{i_0+1} = const$ act on the quadric also. Also, the matrices of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \frac{||\vec{v}||^2}{2} & \vec{v} & 1 \end{pmatrix}$$

induce and preserve the quadric.

The group of affine transformations that acts on the quadric is exactly the Lie group generated by $\mathcal{N}$ and $O(i_0)$. The action is transitive and each of the stabilizer is a conjugate of $O(i_0)$ by elements of $\mathcal{N}$.

The proof of this fact is simply that the such an affine transformation is conjugate to an element a parabolic group in the i-dimensional complete hyperbolic space where the ideal fixed point is identified with $[0, \dots, 0, 1] \in \mathbb{S}^{i_0+1}$ and tangent to the boundary of A^{i_0}.

5.1.2. *Technical lemmas.* Suppose that the semisimple part of $h(\pi_1(S))$ is irreducible for contradiction. Recall $\mathcal{N}_j$ from Equation 16. For $\vec{v} \in \mathbb{R}^{i_0}$, we define

$$\mathcal{N}(\vec{v}) := \left(\begin{array}{cc|cccc} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 & \dots & 0 \\ \vec{0} & 1 & 0 & 0 & \dots & 0 \\ \hline \vec{c}_1(\vec{v}) & v_1 & 1 & 0 & \dots & 0 \\ \vec{c}_2(\vec{v}) & v_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{c}_{i_0+1}(\vec{v}) & \frac{1}{2}||\vec{v}||^2 & v_1 & v_2 & \dots & 1 \end{array}\right) \tag{23}$$

where $||v||$ is the norm of $\vec{v} = (v_1, \cdots, v_i) \in \mathbb{R}^{i_0}$. The elements of our unipotent group $\mathcal{N}$ are of this form since $\mathcal{N}(\vec{v})$ is the product $\prod_{j=1}^{i_0} \mathcal{N}(e_j)^{v_j}$. Note that by the way we defined this, for each k, $k = 1, \dots, i_0$, $\vec{c}_k : \mathbb{R}^{i_0} \to \mathbb{R}^{n-i_0-1}$ are really linear functions of $\vec{v}$ defined as $\vec{c}_k(\vec{v}) = \sum_{j=1}^{i_0} \vec{c}_{kj} v_j$ for $\vec{v} = (v_1, v_2, \dots, v_{i_0})$ so that we form a group. We do not need the property of $\vec{c}_{i_0+1}$ at the moment. We denote by $C_1(\vec{v})$ the $(n - i_0 - 1) \times i_0$-matrix given by the matrix with rows $c_j(\vec{v})$ for $j = 1, \dots, i_0$ and by $c_2(\vec{v})$ the row $(n - i_0 - 1)$-vector $\vec{c}_{i_0+1}(\vec{v})$.

The lower-right $(i_0 + 2) \times (i_0 + 2)$-matrix is form is called the *standard cusp matrix form.*

Since $\mathbb{S}^{i_0}_\infty$ is invariant, it also follows that g, $g \in \Gamma$, is of *standard* form

$$
(24) \qquad \begin{pmatrix} \begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & a_6(g) \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array} \end{pmatrix}
$$

where $S(g)$ is an $(n - i_0 - 1) \times (n - i_0 - 1)$-matrix and $s_1(g)$ is an $(n - i_0 - 1)$-column vector, $s_2(g)$ and $c_2(g)$ are $(n - i_0 - 1)$-row vectors, $C_1(g)$ is an $i_0 \times (n - i_0 - 1)$-matrix, $a_4(g)$ and $a_6(g)$ are i_0-column vectors, $A_5(g)$ is an $i_0 \times i_0$-matrix, $a_8(g)$ is an i_0-row vector, and $a_1(g), a_7(g)$, and $a_9(g)$ are scalars. (We show $a_6(g) = 0$ for any standard form of g soon.)

Denote

$$
\hat{S}(g) = \begin{pmatrix} S(g) & s_1(g) \\ s_2(g) & a_1(g) \end{pmatrix}.
$$

Lemma 5.6. *Let l' in $\mathrm{bd}K$ be a fixed point l' of $\hat{S}(g)$ for an infinite order g, $g \in \mathbf{\Gamma}_E$. Suppose that the leaf l' corresponds to a hemisphere $H^{i_0+1}_{l'}$ where we assume*

$$
(\mathbb{S}^{i_0+1}_{l'} - \mathbb{S}^{i_0+1}_\infty) \cap \mathrm{Cl}(U) \neq \emptyset.
$$

Then $\mathcal{N}$ acts on the open horosphere $U_{l'}$ in $\mathrm{Cl}(U)$ in a component of $H^{i_0+1}_{l'} - \mathbb{S}^{i_0+1}_\infty$.

Proof. The existence of the hemisphere is clear since $\mathbb{S}^{n-i_0-1}$ is considered the space of $(i_0 + 1)$-dimensional hemispheres.

For a fixed point l' of g, only one component $\mathbb{S}^{i_0+1}_{l'} - \mathbb{S}^{i_0+1}_\infty$ meets $\mathrm{Cl}(U)$ since otherwise, $\mathrm{Cl}(K)$ contains a pair of antipodal points.

Each infinite order $g \in \mathbf{\Gamma}_E$ has one attracting fixed point or a repelling fixed point, with norms of the eigenvalue distinct from those in $\mathbb{S}^{i_0}_\infty$. Since g is semisimple in $\mathbb{S}^{n-i_0-1}$ and the fixed point has a distinct eigenvalue from those of $\mathbb{S}^{i_0+1}$, we have a corresponding fixed point in a component above.

Let $A_{l'}$ denote this component. Let $J_{l'} := A_{l'} \cap \mathrm{Cl}(U)$. $\mathcal{N}$ then has the form in $H^{i_0+1}_{l'}$ as

$$
\begin{pmatrix} 1 & 0 & 0 \\ L(\vec{v}^T) & \mathrm{I}_{i_0} & 0 \\ \kappa(\vec{v}) & \vec{v} & 1 \end{pmatrix}
$$

since the $\mathbb{S}^{i_0}_\infty$-part, i.e., the last i_0+1 coordinates, is not changed from one for equation 23 where $L : \mathbb{R}^{i_0} \to \mathbb{R}^{i_0}$ is a linear map. The linearity of L is the consequence of the group property. $\kappa : \mathbb{R}^{i_0} \to \mathbb{R}$ is some function. We consider L as $i_0 \times i_0$-matrix.

If there exists a kernel K_1, then we use $t\vec{v} \in K_1 - \{O\}$ and as $t \to \infty$, we can show that $\mathcal{N}(J_{l'})$ cannot be properly convex.

Also, since $\mathcal{N}$ is abelian, the computations shows that $\vec{v}L\vec{w}^T = \vec{w}L\vec{v}^T$ for every pair of vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^{i_0}$. It follows that L is a symmetric matrix.

Since L hence is nonsingular, we can find a coordinate change of coordinates of $x_{n-i_0+1}, \dots, x_n$ so that $\mathcal{N}$ is now of standard form: We conjugate $\mathcal{N}$ by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for nonsingular A. We obtain

$$\begin{pmatrix} 1 & 0 & 0 \\ AL\vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \kappa(\vec{v}) & \vec{v}A^{-1} & 1 \end{pmatrix}.$$

We thus need to solve for $A^{-1}A^{-1T} = L$, which can be done.

We can factorize each element of $\mathcal{N}$ into forms

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{I}_{i_0} & 0 \\ \kappa(\vec{v}) - \frac{||\vec{v}||^2}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \frac{||\vec{v}||^2}{2} & \vec{v} & 1 \end{pmatrix}.$$

Again, by the group property, $\alpha_7(\vec{v}) := \kappa(\vec{v}) - \frac{||\vec{v}||^2}{2}$ gives us a linear function $\alpha_7 : \mathbb{R}^{i_0} \to \mathbb{R}$. Hence $\alpha_7(\vec{v}) = \kappa_\alpha \cdot \vec{v}$ for $\kappa_\alpha \in \mathbb{R}^{i_0}$. Now, we conjugate $\mathcal{N}$ by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{I}_{i_0} & 0 \\ 0 & -\kappa_\alpha & 1 \end{pmatrix}$$

and this will put $\mathcal{N}$ into the standard form.

Now it is clear that the orbit of $\mathcal{N}(x_0)$ for a point x_0 of $J_{l'}$ is horosphere as we can conjugate so that the first column entries from the second one to the (i_0+1)-th one equals those of the last row. Since $\mathrm{Cl}(U)$ is $\mathcal{N}$-invariant, we obtain that $\mathcal{N}(x_0) \subset J_{l'}$. $\square$

Note that for many of the following lemmas, we do not really need a discreteness assumption on N_K.

Lemma 5.7. *Let $\mathcal{N}$ be an i_0-dimensional nilpotent Lie group with elements of form of equation 23 and K/N_K compact. Then any element $g \in \mathbf{\Gamma}_E$ of form of equation 24 normalizing $\mathcal{N}$ and acting on $\mathbb{S}^{i_0}_\infty$ induces an $i_0 \times i_0$-matrix M_g given by*

$$g\mathcal{N}(\vec{v})g^{-1} = \mathcal{N}(\vec{v}M_g)$$

where M_g is a scalar multiplied by the inverse of an element $O_5(g)$ of a compact Lie group G_E. Actually, we have

- $$M_g = \frac{1}{a_1(g)}(A_5(g))^{-1} = \mu_g O_5(g)^{-1}$$
- $(a_5(g))^2 = a_1(g)a_9(g)$ *or equivalently* $\frac{a_5(g)}{a_1(g)} = \frac{a_1(g)}{a_5(g)}$ *where $a_5(g)$ is defined to be* $|\det(A^5_g)|^{\frac{1}{i_0}}$.
- *Finally, $a_1(g), a_5(g)$, and $a_9(g)$ are all nonzero and we have $a_6(g) = 0$.*

This is true as long as g is in the standard form.

Proof. Since the conjugation by g sends elements of $\mathcal{N}$ to itself in a one-to-one manner, the correspondence between the set of $\vec{v}$ for $\mathcal{N}$ and $\vec{v'}$ is one-to-one.

Since we have $g\mathcal{N}(\vec{v}) = \mathcal{N}(\vec{v}')g$ for vectors $\vec{v}$ and $\vec{v'}$ in $\mathbb{R}^{i_0}$ by the normalization condition, we consider

$$\left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & a_6(g) \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right)\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v}) & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}) & \frac{||\vec{v}||^2}{2} & \vec{v} & 1 \end{array}\right) \tag{25}$$

where $C_1(\vec{v})$ is an $(n-i_0-1) \times i_0$-matrix where each row is a linear function of $\vec{v}$, $c_2(\vec{v})$ is a $(n-i_0-1)$-row vector, $\vec{v}$ is an i_0-row vector, and s is a scalar. This must equal

the following matrix for some $\vec{v'} \in \mathbb{R}$

$$(26) \qquad \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline C_1(\vec{v'}) & \vec{v'}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v'}) & \frac{||\vec{v'}||^2}{2} & \vec{v'} & 1 \end{array}\right)\left(\begin{array}{c|c|c|c} S(g) & s_1(g) & 0 & 0 \\ \hline s_2(g) & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & A_5(g) & a_6(g) \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right).$$

From equation 25, we compute the $(4,3)$-block of the result to be $a_8(g) + a_9(g)\vec{v}$. From Equation 26, the $(4,3)$-block is $\vec{v'}A_5(g) + a_8(g)$. We obtain the relation $a_9(g)\vec{v} = \vec{v'}A_5(g)$ for every $\vec{v}$. Since the correspondence between $\vec{v}$ and $\vec{v'}$ is one-to-one, we obtain

$$(27) \qquad \vec{v'} = a_9(g)\vec{v}(A_5(g))^{-1}$$

for the $i_0 \times i_0$-matrix $A_5(g)$ and we also infer $a_9(g) \neq 0$ and $\det(A_5(g)) \neq 0$. The $(3,2)$-block of the result of Equation 25 equals

$$a_4(g) + A_5(g)\vec{v}^T + \frac{1}{2}||\vec{v}||^2 a_6(g).$$

The $(3,2)$-block of the result of Equation 26 equals

$$(28) \qquad C_1(\vec{v}')s_1(g) + a_1(g)\vec{v}'^T + a_4(g).$$

Since the $||v||^2$-term is only term that is quadratic, we obtain $a_6(g) = 0$ and

$$(29) \qquad A_5(g)\vec{v}^T = C_1(\vec{v}')s_1(g) + a_1(g)\vec{v}'^T.$$

We obtain $a_6(g) = 0$ since $||\vec{v}||^2$ is the only quadratic term.

Let us fix l for the following discussions.

For each g, we can choose a coordinate so that $s_1(g) = 0$ as $\hat{S}(g)$ is semi-simple, which involves the coordinate changes of the first $n - i_0$ coordinate functions only. We have $a_1(g) \neq 0$ because elements of U are moved into ones in U. The coordinate change corresponds to the row operations for the first $(n - i_0)$-rows of the matrices by multiplying the $(n - i_0)$-th row by scalars and subtracting from the higher rows and do the column operations for the first $(n - i_0)$-columns by multiplying the first $(n - i_0 - 1)$ columns by scalars and adding to the $(n - i_0)$-th column. In this coordinate system, the matrix entries of the lower-right $(i_0 + 2) \times (i_0 + 2)$-matrices might change but it will affect the first columns and the first rows. But the matrix form itself will not change. We may also assume that U satisfies $x_{n-i_0} > 0$ since U is convex.

Lemma 5.8. *For $g \in \mathbf{\Gamma}_E - \mathcal{N}$ going to an infinite order element in N_K, we have $\lambda_0(g) \geq \lambda(g)$ and $\lambda_0(g)$ has a fixed point in $\mathbb{S}^{i_0+1}_{l'} - \mathbb{S}^{i_0}_\infty$ where $\{g^j(x)\}_{j\in\mathbb{Z}}$ has an accumulation point $x \in U$ and l' is a leaf corresponding to an attracting or repelling fixed point of $\pi_K(g)$. Thus, we have*

$$(\mathbb{S}^{i_0+1}_{l'} - \mathbb{S}^{i_0}_\infty) \cap \mathrm{Cl}(U) \neq \emptyset.$$

Proof. By Proposition 5.3, we have $\lambda_0(g) \geq \lambda(g)$. Suppose that $\lambda_0(g) = \lambda(g)$. We can assume that $\lambda(g)$ is distinct from $\lambda_{\mathbf{v}_E}$ by the weak uniform middle-eigenvalue condition and taking g^{-1} instead of g if necessary. Let l' denote the fixed point of g in $\mathrm{Cl}(K)$ and hence the corresponding sphere $\mathbb{S}^{i_0+1}_{l'}$ exists in $\mathbb{S}^n$. If there is no such a fixed point in $\mathbb{S}^{i_0+1}_{l'} - \mathbb{S}^{i_0}_\infty$ of g, then we obtain a fixed point y of g in $\mathbb{S}^{i_0}_\infty$ corresponding to $\lambda(g)$ where $\{g^j(x)|j \geq 0\}$ accumulates to $y \in \mathbb{S}^{i_0}_\infty$ for $x \in U$ and y is distinct from $\mathbf{v}_E$ as $\lambda_0(g) > \lambda_{\mathbf{v}_E}(g)$ by the middle-eigenvalue condition. The convex hull of $\mathcal{N}(y)$ is not properly convex in $\mathbb{S}^{i_0}_\infty$ as we can see from the matrix form of $\mathcal{N}$ and some computations as $y \neq \pm\mathbf{v}_E$. Since this is a subset of $\mathrm{Cl}(U)$, we obtained a contradiction.

Suppose that $\lambda_0(g) > \lambda(g)$. Then clearly there exists the great sphere corresponding to the eigenspace of $\lambda_0(g)$ and it is disjoint from $\mathbb{S}^{i_0}_\infty$. Hence, $\{g^j(x)|j \geq 0\}$ accumulates to y for some y that is in this sphere. $\square$

Since $\mathcal{N}$ acts on $\mathbb{S}^{i_0+1}_{l'}$ as a cusp group by Lemma 5.6, it follows that there exists a coordinate change involving the last (i_0+1)-coordinates $x_{n-i_0+1}, \dots, x_n, x_{n+1}$ so that the matrix form of the lower-right $(i_0+2) \times (i_0+2)$-matrix of each element $\mathcal{N}$ is of the standard cusp form. In fact, this is the affine coordinate change making the ellipsoid to be given by the standard quadric. Then each element of $\mathcal{N}$ is of the standard cusp form by Section 5.1.1. So again the change is given by a conjugation by a matrix that differs from I by the lower-right $(i_0+2) \times (i_0+2)$-matrix that also has zero's at the last row and the last column except the last diagonal entry. This is an affine coordinate change when we consider the interior of $H^{i_0+1}_{l'}$ as an affine space. Also, such a coordinate is unique up to the orthogonal transformation fixing the x_{n+1}-direction multiplied by an element of $\mathcal{N}$. This does not change the fact that $s_2(g) = 0$. Let us called $\Phi_{g,l'}$ the adopted coordinate system of the interior $A^{i_0+1}_{l'}$, which depends on g and l'. (Note that the coordinate change does not change the fact that all g are in the standard form with $a_6(g) = 0$.)

Let us use $\Phi_{g,l'}$ for a while using primes for new set of coordinates functions. Under this coordinate system for given g, we obtain $a'_1(g) \neq 0$ and we can recompute to show that $a'_9(g)\vec{v} = \vec{v'}A'_5(g)$ for every $\vec{v}$ as in equation 27. By equation 29 for this case, we obtain

$$\vec{v'} = \frac{1}{a'_1(g)}\vec{v}(A'_5(g))^T. \tag{30}$$

as $s'_1(g) = 0$ here since we are using the coordinate system $\Phi_{g,l'}$. Since this holds for every $\vec{v} \in \mathbb{R}^{i_0}$, we obtain

$$a'_9(g)(A'_5(g))^{-1} = \frac{1}{a'_1(g)}(A'_5(g))^T.$$

Hence $\frac{1}{|\det(A'_5(g))|^{1/i_0}}A'_5(g) \in O(i_0)$. Also,

$$\frac{a'_9(g)}{a'_5(g)} = \frac{a'_5(g)}{a'_1(g)}.$$

Here, $A'_5(g)$ is a conjugate of the original matrix $A_5(g)$ by an affine coordinate change. We say that the matrix of g is in the orthogonal form if $A_5(g)$ is a scalar multiplied by an orthogonal matrix and has the form of the same matrix form as before.

This implies that the original matrix $A_5(g)$ is conjugate to an orthogonal matrix multiplied by a positive scalar for every g. The set of matrices $\{A_5(g)|g \in \mathbf{\Gamma}_E\}$ forms a group since every g is of a standard matrix form (see equation 24) where $a_6(g) = 0$ for every g. Given such a group of matrices normalized to have determinant ± 1, we obtain a compact group G_E by Lemma 5.9.

Since $\mathcal{N}$ acts on the ellipsoid that is in the standard quadratic form, $\mathcal{N}$ is in the cusp form. Finally, we do the above steps again for this coordinate system to obtain the conclusions of the lemma.

□

(Note that it may be $O_5(g)^T \neq O_5(g)^{-1}$ since we only have orthogonality up to conjugation.)

Lemma 5.9. *Suppose that G is a closed subgroup of a linear group $\mathsf{GL}(i_0, \mathbb{R})$ where each element is conjugate to an orthogonal element. Then G is a compact group.*

Proof. Clearly, the norms of eigenvalues of $g \in G$ are all 1. By Fried [26], G is virtually a unipotent group. Since each element is semisimple, we have that G is a subgroup of an orthogonal group under a coordinate system. □

Denote by μ_g the real number $\frac{a_5(g)}{a_1(g)} = \frac{a_9(g)}{a_5(g)}$ for $g \in \mathbf{\Gamma}_E$. We denote by $(C_1(\vec{v}), \vec{v}^T)$ the matrix obtained from $C_1(\vec{v})$ by adding a column vector $\vec{v}^T$.

Lemma 5.10. *Assume as in Lemma 5.7.*

- *K is a cone over a totally geodesic $(n - i_0 - 2)$-dimensional domain K''.*
- *The rows of $(C_1(\vec{v}), \vec{v}^T)$ are proportional to a single vector and we can find a coordinate system where $C_1(\vec{v}) = 0$ and not changing any entries of the lower-right $(i_0 + 2) \times (i_0 + 2)$-submatrices from what we started with.*
- *We can find a common coordinate system where $O_5(g)^{-1} = O_5(g)^T$ and $O_5(g) \in O(i_0)$ for all g and $s_1(g) = s_2(g) = 0$.*
- *In this coordinate system, we have*

$$a_9(g)c_2(\vec{v}) = c_2(\mu_g \vec{v} O_5(g)^{-1})S(g) + \mu_g \vec{v} O_5(g)^{-1} C_1(g). \tag{31}$$

Proof. The assumption implies that $M_g = \mu_g O_5(g)^{-1}$ by Lemma 5.7. We consider the equation

$$g\mathcal{N}(\vec{v})g^{-1} = \mathcal{N}(\mu_g \vec{v} O_5(g)^{-1}). \tag{32}$$

For the second, we consider

$$g\mathcal{N}(\vec{v}) = \mathcal{N}(\mu_g \vec{v} O_5(g))g$$

and consider the lower left $(n - i_0) \times (i_0 + 1)$-matrix of the left side, we obtain

$$\begin{pmatrix} C_1(g) & a_4(g) \\ c_2(g) & a_7(g) \end{pmatrix} + \begin{pmatrix} a_5(g)O_5(g)C_1(\vec{v}) & a_5(g)O_5(g)\vec{v} \\ a_8(g)C_1(\vec{v}) + a_9 c_2(\vec{v}) & a_8(g)\cdot\vec{v}^T + a_9(g)\vec{v}\cdot\vec{v}/2 \end{pmatrix} \tag{33}$$

where the entry sizes are clear. From the right side, we obtain
(34)

$$\begin{pmatrix} C_1(\mu_g \vec{v} O_5(g)^{-1}) & \mu_g O_5(g)^{-1,T} \vec{v}^T \\ c_2(\mu_g \vec{v} O_5(g)^{-1}) & \vec{v}\cdot\vec{v}/2 \end{pmatrix} \hat{S}(g) + \begin{pmatrix} C_1(g) & a_4(g) \\ \vec{v}' \cdot C_1(g) + c_2(g) & a_7(g) + \vec{v}' \cdot a_4(g) \end{pmatrix}.$$

From the top row, we obtain that

(35)

$$(a_5(g) O_5(g) C_1(\vec{v}), a_5(g) O_5(g) \vec{v}^T) = (\mu_g C_1(\vec{v} O_5(g)^{-1}), \mu_g O_5(g)^{-1,T} \vec{v}^T) \hat{S}(g).$$

(36)

$$(a_5(g) C_1(\vec{v}), a_5(g) \vec{v}^T) \hat{S}(g^{-1}) = (\mu_g O_5(g)^{-1} C_1(\vec{v} O_5(g)^{-1}), \mu_g O_5(g)^{-1} O_5(g)^{-1,T} \vec{v}^T).$$

since C_1 is linear where we multiplied the both sides by $O_5(g)^{-1}$. Let us form the subspace V_C in the dual sphere $\mathbb{R}^{n-i_0*}$ spanned by row vectors of $(C_1(\vec{v}), \vec{v}^T)$. Let $\mathbb{S}^*_C$ denote the corresponding subspace in $\mathbb{S}^{n-i_0-1*}$. Then

$$\frac{1}{\det \hat{S}(g)^{\frac{1}{n-i_0-1}}} \hat{S}(g)$$

acts on V_C as a group of bounded linear automorphisms since $O_5(g) \in G$ for a compact group G, and hence $\{\hat{S}(g) | g \in \mathbf{\Gamma}_E\}$ on $\mathbb{S}^*_C$ is in a compact group of projective automorphisms.

We recall that the dual group N_K^* of N_K acts on the properly convex dual domain K^* of K. Then g acts as an element of a compact group.

We claim that $\dim(\mathbb{S}_C) = 0$. Let $\mathbb{S}_M$ be the maximal invariant subspace where each $g \in N_K^*$ acts orthogonally containing $\mathbb{S}_C$. Since N_K^* is semi-simple by Benoist [3], it follows that there exist a complementary subspace $\mathbb{S}^l$ where N_K^* acts on.

Recall that N_K^* acts on the dual K^* of K properly and cocompactly. K^* has an invariant subspace K_1^* and K_2^* so that $K^* = K_1^* * K_2^*$, a join so that $\dim K_1^* = \dim \mathbb{S}_M$ and $\dim K_2^* = \dim \mathbb{S}_C$. We assume that $K_2^* = K^* \cap \mathbb{S}_M$ and $K_2^* \cap \mathbb{S}^l$. Also, by the theory of Benoist [3], N_K^* is isomorphic to $N_{K,1} \times N_{K,2} \times \mathbb{Z}$ where $N_{K,i}$ acts on a properly convex domain K_i^* properly and cocompactly for $i = 1, 2$. But since $N_{K,1}$ acts orthogonally on $\mathbb{S}_M$. The only possibility is that $\dim \mathbb{S}_M = 0$.

Therefore this shows that the rows of $(C_1(\vec{v}), \vec{v}^T)$ are proportional to a single row vector.

Since $(C_1(\vec{e}_j), \vec{e}_j^T)$ has 0 is the last column except for the jth one, it follows that only the j-row is nonzero and moreover, it equals to a scalar multiple of a common vector $(C_1(1, \vec{e}_1), 1)$ for every j where $C_1(1, \vec{e}_1)$ is the first row of $C_1(\vec{e}_1)$. Now we can choose coordinates of $\mathbb{R}^{n-i_0*}$ so that this row vector now has a coordinate $(0, \dots, 0, 1)$. We can also choose so that K_1^* is given by setting the last coordinate be zero. With this change, we need to do conjugation by matrices with the top left $(n - i_0 - 1) \times (n - i_0 - 1)$-submatrix being different from I and the rest of the entries staying the same. This will not affect the expressions of matrices of lower right $(i_0 + 2) \times (i_0 + 2)$-matrices involved here. Thus, $C_1(\vec{v}) = 0$ in this coordinate for all $\vec{v} \in \mathbb{R}^{i_0}$ and $g \in \mathbf{\Gamma}_E - N$.

And the in the above coordinate system, we have $s_1(g) = 0, s_2(g) = 0$. We also gained that K is a join of a point $k = [0, \dots, 0, 1]$ and a domain K'' given by setting $x_{n-i_0} = 0$ in a totally geodesic sphere of dimension $n - i_0 - 2$ by duality.

For the final item we have:

(37)

$$g = \begin{pmatrix} S(g) & 0 & 0 & 0 \\ 0 & a_1(g) & 0 & 0 \\ C_1(g) & a_4(g) & a_5(g)O_5(g) & 0 \\ c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{pmatrix}, \quad \mathcal{N}(\vec{v}) = \begin{pmatrix} \mathrm{I} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \vec{v}^T & \mathrm{I} & 0 \\ c_2(\vec{v}) & \frac{1}{2}||\vec{v}||^2 & \vec{v} & 1 \end{pmatrix}.$$

The normalization of $\mathcal{N}$ shows as in the proof of Lemma 5.7 that $O_5(g)$ is orthogonal now. (See equations 27 and 29.) We consider the lower-right $(i_0 + 1) \times (n - i_0)$-submatrices of $g\mathcal{N}(\vec{v})$ and $\mathcal{N}(\vec{v'})g$. For the first one, we obtain

$$\begin{pmatrix} C_1(g) & a_4(g) \\ c_2(g) & a_7(g) \end{pmatrix} + \begin{pmatrix} a_5(g)O_5(g) & 0 \\ a_8(g) & a_9(g) \end{pmatrix} \begin{pmatrix} 0 & \vec{v}^T \\ c_2(\vec{v}) & \frac{1}{2}||\vec{v}||^2 \end{pmatrix}$$

For $\mathcal{N}(\vec{v'})g$, we obtain

$$\begin{pmatrix} 0 & \vec{v'}^T \\ c_2(\vec{v'}) & \frac{1}{2}||\vec{v'}||^2 \end{pmatrix} \begin{pmatrix} S(g) & 0 \\ 0 & a_1(g) \end{pmatrix} + \begin{pmatrix} \mathrm{I} & 0 \\ \vec{v'} & 1 \end{pmatrix} \begin{pmatrix} C_1(g) & a_4(g) \\ c_2(g) & a_9(g) \end{pmatrix}$$

Considering $(2, 1)$-blocks, we obtain

$$c_2(g) + a_9(g)c_2(\vec{v}) = c_2(\vec{v'})S(g) + +\vec{v'}C_1(g) + c_2(g).$$

□

We consider the case when M_g for all g are in a compact group; i.e., with all scalar factor $= \pm 1$.

Lemma 5.11. *Assume as in Lemma 5.7. Suppose additionally that every $g \in \Gamma \to M_g$ is so that M_g is in a fixed compact group $O(i_0)$. Then we can find coordinates so that the following holds for all g.*

- $O_5(g)^{-1}a_4(g) = (a_8(g))^T$ *or* $a_4(g)^T O_5(g) = a_8(g)^T$.
- $a_1(g) = a_9(g) = \lambda_{\mathbf{v}_E}(g)$ *and* $A_5(g) = \lambda_{\mathbf{v}_E}(g)O^5_g$.

Proof. Since $M_g = O_5(g)^{-1}$, we have $\mu_g = 1$ and $a_1(g) = a_9(g) = a_5(g) = \lambda_{\mathbf{v}_E}(g)$ and $A_5(g) = \lambda_{\mathbf{v}_E}(g)O_5(g)$ by Lemma 5.7.

Again, we use equations 25 and 26 with the gained knowledge. We need to only consider lower right $(i_0+2)\times(i_0+2)$-matrices.

$$\begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g) & a_5(g)O_5(g) & 0 \\ a_7(g) & a_8(g) & a_9(g) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \vec{v}^T & I & 0 \\ \frac{1}{2}||\vec{v}||^2 & \vec{v} & 1 \end{pmatrix} \tag{38}$$

$$= \begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g) + a_5(g)O_5(g)\vec{v}^T & a_5(g)O_5(g) & 0 \\ a_7(g) + a_8(g)\vec{v}^T + \frac{a_9(g)}{2}||\vec{v}||^2 & a_8(g) + a_9(g)\vec{v} & a_9(g) \end{pmatrix}. \tag{39}$$

This equals

$$\begin{pmatrix} 1 & 0 & 0 \\ \vec{v'}^T & I & 0 \\ \frac{1}{2}||\vec{v'}||^2 & \vec{v'} & 1 \end{pmatrix} \begin{pmatrix} a_1(g) & 0 & 0 \\ a_4(g) & a_5(g)O^5_g & 0 \\ a_7(g) & a_8(g) & a_9(g) \end{pmatrix} \tag{40}$$

$$= \begin{pmatrix} a_1(g) & 0 & 0 \\ a_1(g)\vec{v'}^T + a_4(g) & a_5(g)O_5(g) & 0 \\ \frac{a_1(g)}{2}||\vec{v'}||^2 + \vec{v'}a_4(g) + a_7(g) & a_5(g)\vec{v'}O_5(g) + a_8(g) & a_9(g) \end{pmatrix}. \tag{41}$$

Then by comparing the $(3,2)$-blocks, we obtain $a_8(g)+a_9(g)\vec{v} = a_8(g)+a_5(g)\vec{v'}O_5(g)$. Thus, $\vec{v} = \vec{v'}O_5(g)$

From the $(3,1)$-blocks, we obtain

$$a_1(g)\vec{v'}\cdot\vec{v'}/2 + \vec{v'}a_4(g) = a_8(g)\vec{v}^T + a_9(g)\vec{v}\cdot\vec{v}/2.$$

Since the quadratic forms have to equal each other, we obtain $\vec{v}O_5(g)^{-1}\cdot a_4(g) = \vec{v}\cdot a_8(g)$ for all $\vec{v}$. Thus, $(O_5(g)^T a_4(g))^T = a_8(g)^T$. □

Thus, in cases that we are concerned in this section, and by taking a finite index subgroup of Γ, we conclude that each $g \in \Gamma - N$ has the form

$$
\left(\begin{array}{c|c|c|c}
S(g) & 0 & 0 & 0 \\
\hline
0 & \lambda_{\mathbf{v}_E}(g) & 0 & 0 \\
\hline
C_1(g) & \lambda_{\mathbf{v}_E}(g)\vec{v}_g^T & \lambda_{\mathbf{v}_E}(g)O_5(g) & 0 \\
\hline
c_2(g) & a_7(g) & \lambda_{\mathbf{v}_E}(g)\vec{v}_g O_5(g) & \lambda_{\mathbf{v}_E}(g)
\end{array}\right). \tag{42}
$$

Lemma 5.12. *If g of form of equation 42 centralizes a Zariski dense subset A' of $\mathcal{N}$, then $O_5(g) = \mathrm{I}_{i_0}$.*

Proof. Note that the subset A'' of $\mathbb{R}^i$ corresponding to A' is also Zariski dense in $\mathbb{R}^i$. $g\mathcal{N}(\vec{v}) = \mathcal{N}(\vec{v})g$ shows that $\vec{v} = \vec{v}O_5(g)$ for all $\vec{v} \in A''$. Hence $O_5(g) = \mathrm{I}$. □

Proposition 5.13. *Assume as in Lemma 5.7. Suppose additionally that every $g \in \Gamma \to M_g$ is so that M_g is in a fixed compact group $O(i_0)$. Assume also that $\mathbf{\Gamma}_E$ satisfies the weak uniform middle-eigenvalue conditions and it normalizes and virtually centralizes $\mathcal{N}$. Suppose that N_K acts properly discontinuously, discretely and cocompactly on K. Then there is a projective embedding of K'' in the closure of $\partial\tilde{\mathcal{O}}$ invariant under $\mathbf{\Gamma}_E$, and one can find a coordinate system so that for every $\mathcal{N}(\vec{v})$ and each element g of $\mathbf{\Gamma}_E$ are written so that*

- $C_1(\vec{v}) = 0, c_2(\vec{v}) = 0$, *and*
- $C_1(g) = 0$ *and* $c_2(g) = 0$.

Proof. Let $\mathbf{\Gamma}'_E$ denote the finite index subgroup of $\mathbf{\Gamma}_E$ centralizing $\mathcal{N}$ and a product of cyclic and hyperbolic groups.

By Lemma 5.10, K is a join $k * K''$ for a point $k \in \mathrm{Cl}(K)$ and an open convex domain K'' of dimension $n - i_0 - 2$ in $\mathrm{bd}K$. Since K/N_K is compact, K has a compact set F which every orbit meets. K is foliated by open lines from a point $k \in \mathrm{Cl}(K)$ to points of K''. That is, K is the interior of the join $k * \mathrm{Cl}(K'')$. Call these k-radial lines. Take such a line line l and a sequence of points $k_m \to k_\infty \in K''$ as $m \to \infty$. Hence, there exists a sequence $\{\gamma_m\}$ of elements of Γ so that $\gamma_m(k_m) \in F$ and $\gamma_m(l)$ is a line passing F so that $\gamma_m(\partial_1 l) \to k_\infty$ for the endpoint $\partial_1 l$ of l in K''. Since K'' is properly convex, this implies $\{\gamma_m | K''\}$ is a bounded sequence of transformations and hence γ_m is of form:

$$
\left(\begin{array}{c|c|c|c}
\delta_m O_m & 0 & 0 & 0 \\
\hline
0 & \lambda_m & 0 & 0 \\
\hline
C_{1,m} & \lambda_m \vec{v}_m^T & \lambda_m O_5(\gamma_m) & 0 \\
\hline
c_{2,m} & a_7(\gamma_m) & \lambda_m \vec{v}_m O_5(\gamma_m) & \lambda_m
\end{array}\right) \tag{43}
$$

where O_m is a bounded sequence of matrices in $\mathbf{Aut}(K'')$ in $\mathsf{SL}(n-i_0-1, \mathbb{R})$ since the set of projective automorphisms of K'' moving points only bounded distances is bounded.

We note that $\delta_m^{n-i_0-1}\lambda_m^{i_0+2} = 1$ and $\delta_m/\lambda_m \to 0$ as $\gamma_m|l$ pushes the points towards the vertex k of K. For the chosen single element γ_m, we can find the subspace $\mathbb{S}^{n-i_0-1}$ containing K'' and $\mathbf{v}_E, \mathbf{v}_{E-}$ where γ_m acts on by the $(n-i_0) \times (n-i_0)$-matrix of form

$$\begin{pmatrix} \delta_m O_m & 0 \\ & \\ 0 & \lambda_m \end{pmatrix}.$$

We choose m so that the norms of eigenvalues of $\delta_m O_m$ is strictly much smaller than the norm of λ_m, the unique norm of the eigenvalues of the lower-right $(i_0+2) \times (i_0+2)$-matrix. We fix one such m_0. Let $S(K''_{m_0})$ denote the γ_{m_0}-invariant subspace corresponding to subspaces associated with the real sum of the Jordan-block subspaces with norms of eigenvalues $< \lambda_{m_0}$. We choose a coordinate system of $\mathbb{S}^n$ so that γ_{m_0} is of form so that $C_{1,m_0} = 0, c_{2,m_0} = 0$. Then there exists a compact proper convex domain K''_{m_0} in $S(K''_{m_0})$ mapping to $\mathsf{Cl}(K'')$ under under the projection Π_K.

We will now show that K'''_{m_0} is invariant under $\mathbf{\Gamma}_E$: let n_1 be an element of $\mathcal{N}$. Then $\gamma_{m_0} n_1 = n_1 \gamma_{m_0}$. Since $n_1\gamma_{m_0}(S(K_{m_0})) = \gamma_{m_0} n_1(S(K_{m_0}))$, we have $n_1(S(K_{m_0})) = \gamma_{m_0} n_1(S(K_{m_0}))$. Since the form of γ_{m_0} determines $S(K_{m_0})$ using span of Jordan-block subspaces, and $n_1(S(K_{m_0}))$ is a γ_{m_0}-invariant subspace, it follows that $n_1(S(K_{m_0})) = S(K_{m_0})$ for all $n_1 \in \mathcal{N}$. Now it is easy to see that $n_1(K_{m_0}) = S(K_{m_0})$ for all $n_1 \in \mathcal{N}$ since they map to K_{m_0} under Π_K. Hence, it follows that $C_1(\vec{v}) = 0$ and $c_2(\vec{v}) = 0$ for every $\vec{v} \in \mathbb{R}^i$ in this system of coordinates.

Let $B(K_{m_0})$ denote the tube that is a union of rays passing K_{m_0}. Let $S(K_{m_0})$ denote the minimal subspace of $\mathbb{S}^n$ containing $B(K_{m_0})$. Then $S(K_{m_0})$ is the unique subspace that $\mathcal{N}$ acts fixing every point. Let $B(\Omega_E)$ denote the tube with vertex $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ corresponding to directions of Ω_E. We note that $B(K_{m_0}) = B(\Omega_E) \cap S(K_{m_0})$.

For any element $g \in \mathbf{\Gamma}'_E$, we also have $\mathcal{N}(\vec{v})g = g\mathcal{N}(\vec{v})$ for all $\vec{v} \in \mathbb{R}^i$. Again we have $\mathcal{N}(\vec{v})g(x) = g(x)$ for all $x \in K_{m_0}$ and $\vec{v} \in \mathbb{R}^i$. This implies that

$$g(K_{m_0}) \subset S(K_{m_0}) \cap B(\mathsf{Cl}(\Omega_E)) = B(K_{m_0})$$

again since $S(K_{m_0})$ is the unique fixed subspace of $\mathcal{N}(\vec{v})$ for all $\vec{v}$. Therefore, $\mathbf{\Gamma}'_E$ acts on $B(K_{m_0})$.

Each irreducible hyperbolic factor group $\mathbf{\Gamma}_i \cap \mathbf{\Gamma}'_E$ satisfies the uniform middle-eigenvalue conditions. Thus, it acts on an invariant set $K_i \subset \mathsf{Cl}(K) \subset \mathbb{S}^{n-i_0-1}$. Here, we don't consider one corresponding to the vertex k in K and hence the irreducible invariant subspaces are all in the complimentary subspace of k in $K \subset \mathbb{S}^{n-i_0-1}$. It follows that there exists a $\mathbf{\Gamma}_i$-invariant set K'_i in $B(K_{m_0})$ distanced from $\mathbf{v}_E$ and $\mathbf{v}_{E-}$, which is unique if $\mathbf{\Gamma}_i$ is hyperbolic by Theorem 3.5. If $\mathbf{\Gamma}_i$ is not hyperbolic, then it is a trivial group, K_i is a point, and K'_i is not unique. $\mathbf{\Gamma}'_E$ acts each segment $B(K_i)$ corresponding to the 0-dimensional K_i either trivially or with a unique common fixed point by the commutative nature of the center and factor groups. We choose the unique fixed point or the arbitrary fixed point the segment corresponding to K_i. The join of $K'_1, \dots, K'_{l_0}$ in $B(K_{m_0})$ is a totally geodesic and $\mathbf{\Gamma}'_E$-invariant. Therefore, K_{m_0} is exactly this join.

Since $\mathbf{\Gamma}_E/\mathbf{\Gamma}'_E$ is finite, there are finitely many sets of form $g(K_m)$ for $g \in \mathbf{\Gamma}_E$. If they are not identical, there exists at least one g' so that $g'(K_m) \neq K_m$. The action of $\gamma_m^i(g'(K_m))$ then produces infinitely many distinct sets of form $g(K_m)$. This is a contradiction. Hence $g(K_m) = K_m$ for all $g \in \mathbf{\Gamma}_E$. This implies that $c_2(\vec{v}) = 0$, and $C_1(g) = 0$ and $c_2(g) = 0$.

□

We remark that Propositions 5.13 and 5.26 have very similar proof. The first one is much simpler, and so we wrote both proofs. It seems worth repeating the proof for convincing the readers.

5.2. **Joins and quasi-joins.** We continue to assume as in Lemma 5.7. Suppose additionally that every $g \in \Gamma \to M_g$ is so that M_g is in a fixed compact group $O(i_0)$. Thus, $\mu_g = 1$ identically.

Now we let K'' denote the interior of K_{m_0} obtained above. Let us consider the subspaces $\mathbb{S}^{n-i_0-1}$ containing $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ and K'' and the transversal geodesic subspace $\mathbb{S}^{i_0+1}$ containing $\mathbb{S}^{i_0}_\infty$ meeting the first sphere at $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$ and corresponding to the vertex k above. We consider $\mathbf{v}_E$ to have coordinates $[0, \dots, 0, 1]$. We have a group $\mathcal{N}$ acting on these two subspaces fixing $\mathbb{S}^{n-i_0-2}$ in $\mathbb{S}^{n-i_0-1}$ not meeting $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$. $\mathbb{S}^{n-i_0-2}$ contains the standard points $[e_i]$ for $i = 1, \dots, n - i_0 - 1$. and $\mathbb{S}^{i_0+1}$ contains $[e_i]$ for $i = n - i_0, \dots, n + 1$.

Let H be the n-hemisphere defined by $x_{n-i_0} > 0$. Then by convexity of U, we can choose H so that $K'' \subset H$ and $\mathbb{S}^{i_0}_\infty \subset \mathrm{Cl}(H)$.

We consider a group $\mathcal{N}$ having the form of equation 23 with $C_1(\vec{v}) = 0, c_2(\vec{v}) = 0$ and the group G of form of equation 42 with $s_1(g) = 0$, $s_2(g) = 0$, and $O_5(g) = \mathrm{I}_{i_0}$, $C_1(g) = 0$ and $c_2(g) = 0$. We will assume that $G \cap \mathcal{N}$ is a lattice in $\mathcal{N}$. Computations will show that $\mathcal{N}$ is centralized by G. G acts on a properly convex set K in $\mathbb{S}^{n-i_0-1}$ so that $\mathrm{Cl}(K)$ equals a join $k * K''$ for k corresponding to $\mathbb{S}^{i_0+1}$. (Recall the projection $\mathbb{S}^n - \mathbb{S}^{i_0}_\infty$ to $\mathbb{S}^{n-i_0-1}$.)

We define invariants from the form of equation 42

$$\alpha_7(g) := \frac{a_7(g)}{\lambda_{\mathbf{v}_E}(g)} - \frac{||\vec{v}_g||^2}{2}$$

for every $g \in G$. Note that $\alpha_7(g^n) = n\alpha_7(g)$ and $\alpha_7(gh) = \alpha_7(g) + \alpha_7(h)$ whenever $g, h, gh \in G$.

Here $\alpha_7(g)$ is determined by factoring the lower-right $(i_0+2)\times(i_0+2)$-matrix of the matrix of g into commuting matrices of form

(44)

$$\left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & 0 & \mathrm{I}_{i_0} & 0 \\ \hline 0 & \alpha_7(g) & \vec{0} & 1 \end{array}\right)\left(\begin{array}{c|c|c|c} S_g & 0 & 0 & 0 \\ \hline 0 & \lambda_{\mathbf{v}_E}(g) & 0 & 0 \\ \hline 0 & \lambda_{\mathbf{v}_E}(g)\vec{v}_g & \lambda_{\mathbf{v}_E}(g)O_5(g) & 0 \\ \hline 0 & \lambda_{\mathbf{v}_E}(g)\frac{||\vec{v}||^2}{2} & \lambda_{\mathbf{v}_E}(g)\vec{v}_g O_5(g) & \lambda_{\mathbf{v}_E}(g) \end{array}\right).$$

We define G_+ to be a subset of G consisting of elements g so that the largest norm $\lambda_1(g)$ of the eigenvalue occurs at the vertex k. Then since $\mu_g = 1$, we necessarily have $\lambda_1(g) = \lambda_{\mathbf{v}_E}(g)$ with all other norms of the eigenvalues occurring at K'' is strictly less than $\lambda_{\mathbf{v}_E}(g)$. The second largest norm $\lambda_2(g)$ of the eigenvalue occurs at the complementary subspace K'' of k in $\mathrm{Cl}(K)$. Thus, G_+ is a semigroup. The condition that $\alpha_7(g) \geq 0$ for $g \in G_+$ is said to be the *positive translation condition.*

We define the *size* of a subset A of $\mathbb{S}^n$ to be

$$\sup\{\mathbf{d}(\mathbf{v}_E, x) | x \in A\}.$$

Notice that the first matrix also extends to automorphisms of $\mathbb{S}^n$ acting on each hemisphere with boundary $\mathbb{S}^{i_0}_\infty$. We explain this. Let l correspond to a leaf in K. $\mathcal{N}$ acts on an ellipsoid E_l in any i_0+1-dimensional sphere l passing $\mathbf{v}_E$ containing $\mathbb{S}^{i_0}_\infty$. Then g sends E_l to an ellipsoid $g(E_l)$ in $g(l)$. It is easy computation to show that the size of $g(E_l)$ depends on the number. If the number goes to $-\infty$, then $\{g(E_i)\}$ converges to a hemisphere.

Again, we define $\mu(g) := \frac{\alpha_7(g)}{\log \frac{\lambda_{\mathbf{v}_E}(g)}{\lambda_2(g)}}$ where $\lambda_2(g)$ denote the second largest norm of the eigenvalues of g and we restrict $g \in G_+$. The condition $\mu(g) > C_0, g \in G_+$ for a uniform constant C_0 is called the uniform positive translation condition.

Suppose that G is an end fundamental group.

Proposition 5.14. *Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\mathbf{\Gamma}_E$ be the end fundamental group. Let E be a convex but not properly convex end and G acts on a properly convex domain discretely on the end neighborhood U fixing $\mathbf{v}_E$, and it satisfies the weakly uniform middle-eigenvalue condition. Suppose that elements of G and $\mathcal{N}$ are of form of equation 37 with*

$$C_1(\vec{v}) = 0, c_2(\vec{v}) = 0, C_1(g) = 0, c_2(g) = 0$$

for every $\vec{v} \in \mathbb{R}^{i_0}$ and $g \in G$. Let G be an end fundamental group acting on a properly convex end neighborhood U.

- *The positive translation condition $\alpha_7 \geq 0$ is a necessary condition that G acts on a properly convex domain in H.*

- *The uniform positive translation condition is equivalent to the condition that G acts on a closed properly convex end neighborhood U' whose closure meets $\mathbb{S}^{i_0+1}_l$ at $\mathbf{v}_E$ only.*
- *If G does not satisfy the uniform positive translation condition, then U is a join and α_7 is identically zero.*
- *α_7 is identically zero, then U is a join.*

Proof. Suppose that $\alpha_7(g) < 0$ for some $g \in G_+$. Then the action by g gives us $g^n(E_l)$ growing exponentially and hence $\{g^n(E_l)\}$ converges geometrically to an (i_0+1)-dimensional hemisphere. Thus, G cannot act on a properly convex domain.

Suppose that G acts with a uniform positive translation condition. Given a point $x = [\vec{v}] \in \mathbb{S}^n$ where $\vec{v} = \vec{v}_s + \vec{v}_h$ where $\vec{v}_s$ is in the direction of K'' and $\vec{v}_h$ is in one of H^{i_0+1}. If $g \in G_+$, then $g[\vec{v}] = [g\vec{v}_s + g\vec{v}_h]$ where $[g\vec{v}_s] \in K''$ and $[g\vec{v}_h] \in H^{i_0}_l$. We can easily show that the Euclidean length of $g\vec{v}_s$ is decreased compared to v_s. and that of $g\vec{v}_h$ is increased.

Let $g_i \in G_+$ be a sequence so that $\log\frac{\lambda_1(g_i)}{\lambda_2(g_i)} \to \infty$ We also can compute that $[g_i\vec{v}_h] \to \mathbf{v}_E$ since the uniform positive translation condition implies that $\alpha_7(g_i) \to \infty$. Hence, $g_i(x) \to \mathbf{v}_E$ if $\log\frac{\lambda_1(g_i)}{\lambda_2(g_i)} \to \infty$.

By similar arguments $g_i^{-1}(x) \to K''$ if $\log\frac{\lambda_1(g_i)}{\lambda_2(g_i)} \to \infty$.

Choose an element $g \in G_+$ so that $\lambda_1(g) > \lambda_2(g)$ and let F' be the fundamental domain in K. This corresponds to a radial subset F bounded away at a distance from k and K'' in U. The set F' has the property that $|\log\frac{\lambda_1(g')}{\lambda_2(g')}| < C_F$ for a positive constant whenever $g'(x_0) \in F$ for a fixed $x_0 \in F$.

Let $G_F := \{g \in G | g'(x_0) \in F\}$. $|\log\frac{\lambda_1(g)}{\lambda_2(g)}|$ is uniformly bounded by a number C_0 and $g \in G - G_+$, such an element g is an element $g_1 \in G_+$ with $g = hg_1$ and h has a uniformly bounded word length depending on C_0. Hence, $\alpha_7(g)$ is bounded below by some negative number since $\alpha_7(g_1) > 0$. There is an upper bound on the size of $g(H_l \cap U)$. In the affine coordinates this means that there is a lower bound on values of linear function obtained from x_{n+1}.

Thus, the convex hull C_F of $\bigcup_{g' \in G_F} g'(H_l)$ is a properly convex set because of this: Note that $\{g^i(C_F)\} \to \mathbf{v}_E$ for $i \to \infty$ and $\{g^i(C_F)\} \to K''$ for $i \to -\infty$. Thus, the convex hull of these sets is properly convex also since they are uniformly bounded from $\mathbf{v}_E$: We can think of this set as product of a bounded convex set equivalent to K multiplied by a complete affine space of dimension $i_0 + 1$ in an affine space given by H^o. Each of $g(H_l \cap U)$ is given by

$$x_{n+1} \geq x_{n-i_0}^2 + \cdots + x_{n+1}^2 + C_g$$

since $\mathcal{N}$ acts on each. C_g has a lower bound say C_U. Then we can find a finitely many supporting half-spaces whose intersection is properly convex. (For example, sufficiently large translations of supporting hyperplanes of form

$$a_{n-i_0}x_{n-i_0} + \cdots + a_{n+1}x_{n+1} = C'$$

for each equation are enough.) Therefore, it is clear that the convex hull of $\{g(E_l) | g \in G\}$ is properly convex.

Let U' be this properly convex hull end neighborhood of $\mathbf{v}_E$. Suppose $\mathrm{Cl}(U')$ meets $\mathbb{S}^{i_0+1}_l - \{\mathbf{v}_E\}$. Let x' be the point. The orbit $\mathcal{N}(x')$ is an ellipsoid in $\mathbb{S}^{i_0+1}_l$ in $\mathrm{Cl}(U')$. If $\alpha_7(g) \neq 0$, we may choose g or g^{-1} so that $\alpha_7 > 0$. Then acting by $g^i(\mathcal{N}(x'))$ becomes larger and larger and geometrically converges to a hemisphere $H^{i_0+1}_l$ as $i \to \infty$ or $i \to -\infty$, a contradiction. Thus, it follows that $\mathrm{Cl}(U') \cap \mathbb{S}^{i_0+1}_l = \{\mathbf{v}_E\}$.

Conversely, suppose that G acts on a properly convex end neighborhood U'.

Recall that a real line is a complement of a point of $\mathbb{RP}^1$. Let z_1 be a real valued projective function on K with $z_1 = 0$ on K'' and $z_1 = \infty$ on the vertex k. This induces a real valued projective function on U also by precomposing with Π_K. Let us choose a fundamental domain F as above. An element $g \in G$ with the property that $\left|\log \frac{\lambda_{K''}(g)}{\lambda_k(g)}\right| < C_F$ where $\lambda_{K''}(g)$ is the largest norm of the eigenvalue of g in K'' and $\lambda_k(g)$ denote the eigenvalue at k has the property that $g(x_0)$ is in the region given by z_1-value between $1/C_2$ and C_2 for a positive constant C_2 depending only on C_F and we assume $z_1(x_0) = 1$. It follows that $g^{-1}(x_0)$ is in the region given by the z_1-value between $1/C_2$ and C_2 also. Let G_C denote this subset of the elements.

Suppose that there exists a sequence $g_i \in G$ with above property so that $\alpha_7(g_i) \to \infty$, then $\{g_i(E_l)\}$ becomes larger and larger and every convergent subsequence converges to a hemisphere geometrically. Since U is properly convex, this cannot happen. Thus, $\{\alpha_7(g)|g \in G_C\}$ is bounded below. Similarly $\{\alpha_7(g)|g \in G_C\}$ is bounded above as we can use $\alpha_7(g^{-1})$.

For a element $g \in G_+$, we take the fundamental domain F of U for g containing $x_0 \in K$. Then we see that the subset $\{g|g(x_0) \in F\}$ is a subset of $G_{C'_F}$ for some constant C'_F. Now, g and $G_{C'_F}$ generate G. Each element g' of G is in $g^i G_{F'_F}$ for some i and hence we obtain

$$-C \leq |\alpha_7(g') - i\alpha_7(g)| \leq C.$$

The uniform positive translation condition is now easily be shown since we need to consider the single element g and its powers.

$\square$

The second case E is said to be *quasi-joined end* and G now is called a *quasi-joined end group.*

Remark 5.15. We give a bit more explanations. Recall that the space of segments in a hemisphere H^{i_0+1} with the vertex $\mathbf{v}_E, \mathbf{v}_{E-}$ form an affine space A^i one-dimension lower, and the group $\mathbf{Aut}(H^{i_0+1})_{\mathbf{v}_E}$ of projective automorphism of the hemisphere fixing $\mathbf{v}_E$ maps to $\mathbf{Aff}(A^{i_0})$ with kernel K equal to transformations of an $(i_0+2) \times (i_0+2)$-matrix form

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & \mathrm{I}_{i_0} & 0 \\ \hline b & \vec{0} & 1 \end{array}\right) \tag{45}$$

where $\mathbf{v}_E$ is given coordinates $[0, 0, \dots, 1]$ and a center point of $H_l^{i_0+1}$ the coordinates $[1, 0, \dots, 0]$. In other words the transformations are of form

$$[1, x_1, \dots, x_{i_0}, 1] \to [1, x_1, \dots, x_{i_0}, 1 + b] \tag{46}$$

and hence b determines the kernel element.

5.2.1. *Splitting the ends.*

Theorem 5.16. *Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a topologically tame properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\mathbf{\Gamma}_E$ be the end fundamental group, and it satisfies the weakly uniform middle-eigenvalue condition. The there exists a finite cover $\Sigma_{E'}$ of Σ_E so that E' is a join or a quasi-join of a totally geodesic ends and a cusp end. The end fundamental group is virtually a product of that of a lens-type end and a cusp end and an infinite cyclic group in the center.*

Proof. By Lemma 5.7, $h(g)\mathcal{N}(\vec{v})h(g)^{-1} = \mathcal{N}(\vec{v}M_g)$ where M_g is a scalar multiplied by an element of a copy of an orthogonal group $O(i)$.

Since $N \subset \mathcal{N}$ is a discrete cocompact, it follows that N is virtually isomorphic to $\mathbb{Z}^{i_0}$. Without loss of generality, we assume that N is a cocompact subgroup of $\mathcal{N}$. $h(g)Nh(g)^{-1} = N$. Since N corresponds to a lattice $L \subset \mathbb{R}^n$ by the map $\mathcal{N}$, and the conjugation by $h(g)$ corresponds to a map given by right multiplication $M_g : L \to L$. Thus, $M_g : L \to L$ has to be isomorphic to an element of $\mathsf{SL}(i_0, \mathbb{Z})$ and is orthogonal. Hence, the image of $g \in h(\pi_1(E)) \mapsto M_g \in \mathsf{SL}(i_0, \mathbb{Z}))$ is a finite order group. Thus, $\mathbf{\Gamma}_E$ has a finite index group $\mathbf{\Gamma}'_E$ centralizing $\mathcal{N}$.

We find a kernel K_1 of this map and take $\Sigma_{E'}$ to be the corresponding cover of Σ_E. By Proposition 5.13, we have the result. Now γ_m is in the center since $M_g = \mathsf{I}$.

Since $\mathbf{\Gamma}_E$ acts on a properly convex set with an appropriate matrix form for each element, Proposition 5.14 implies that $\mathbf{\Gamma}_E$ satisfies the positive translation condition. Proposition 5.14(i) and (ii) imply that $\mathbf{\Gamma}_E$ is either a join or a quasi-join group. □

5.3. **Indiscrete case.** Let Σ_E be the end orbifold of a nonproperly convex and radial end E of a topologically tame properly convex n-orbifold $\mathcal{O}$ with radial ends. Let $\mathbf{\Gamma}_E$ be the end fundamental group. Let U be an end neighborhood in $\tilde{\mathcal{O}}$ corresponding to an end vertex $\mathbf{v}_E$.

We can assume that ∂U is smooth by smoothing if necessary.

In the above Ω_E fibers over a properly convex domain $K \subset \mathbb{S}^{n-i_0-1}$ with fibers open hemispheres of dimension i_0. Denote by π_E the projection to K.

We have an exact sequence

$$1 \to N \to \pi_1(E) \xrightarrow{\pi_K} N_K \to 1$$

Denote by $\mathbf{\Gamma}_E$ the holonomy group in $\mathsf{SL}(n+1, \mathbb{R})$ of group $\pi_1(E)$.

An element $g \in \mathbf{\Gamma}_E$ is of form:

$$g = \left(\begin{array}{c|c} K(g) & 0 \\ \hline * & U(g) \end{array}\right). \tag{47}$$

Here $K(g)$ is an $(n-i_0)\times(n-i_0)$-matrix and $U(g)$ is in $(i_0+1)\times(i_0+1)$-matrix acting on $\mathbb{S}^{i_0}_\infty$. We note $\mathbf{dev} K(g)\det U(g) = 1$.

5.4. **Estimations with $KA\mathbb{U}$.** Let $\mathbb{U}$ denote the maximal unipotent subgroup of $\mathbf{Aut}(\mathbb{S}^n)_{\mathbb{S}^{i_0}_\infty}$ given by lower triangular matrices with diagonals equal to 1.

Lemma 5.17. *The matrix of $g \in \mathbf{Aut}(\mathbb{S}^n)$ can be written under a coordinate system orthogonal at V^{i_0+1} as $k(g)a(g)n(g)$ where $k(g)$ is an element of $O(n+1)$, $a(g)$ is a diagonal element, and $n(g)$ is in the group $\mathbb{U}$ of unipotent lower triangular matrices. Also, diagonal elements of $a(g)$ are the norms of eigenvalues of g as elements of $\mathbf{Aut}(\mathbb{S}^n)$.*

Proof. Let $\vec{v_1}, \dots, \vec{v_{i_0+1}}, \vec{v}_{i_0+2}, \dots, \vec{v}_{n+1}$ denote the basis vectors of $\mathbb{R}^{n+1}$ that are chosen from the Jordan-block subspaces of g with the same norms of eigenvalues where $\vec{v_j} \in V^{i_0+1}$ for $j = 1, \dots, i_0+1$. We require $[\vec{v_1}] = \mathbf{v}_E$.

Now we fix a Euclidean metric on $\mathbb{R}^{n+1}$. We obtain vectors $\vec{v_1}, \dots, \vec{v_{i_0+1}}, \vec{v}_{i_0+2}, \dots, \vec{v}_{n+1}$ by the Gram-Smidth orthonormalization process using the corresponding Euclidean metric on $\mathbb{R}^{n+1}$. Then the desired result follows by writing the matrix of g in terms of coordinate given by letting the basis vectors $\vec{v}_i' = \vec{u}_{n+1-i}$. (See also Proposition 2.1 of Kostant [40].)

□

Let $\lambda_0(g)$ and $\lambda_{n+1}(g)$ denote the largest and the smallest norm of eigenvalues of g not occurring for vectors in $\mathbb{S}^{i_0}_\infty$. Let $\lambda_1(g)$ and $\lambda_{i_0+1}(g)$ denote the largest norm and the smallest norm of eigenvalues of g restricted to V^{i_0+1}.

We define

$$\mathbb{U}' := \bigcup_{k\in O(n+1)} k\mathbb{U}k^{-1}.$$

Corollary 5.18. *Suppose that we have for a positive constant C_1, and $g \in \mathbf{\Gamma}_E$,*

$$\frac{1}{C_1} \leq \lambda_n(g), \lambda_0(g) \leq C_1.$$

Then g is in a bounded distance from $\mathbb{U}'$ with the bound depending only on C_1.

Proof. By Lemma 5.17, we can find an element $k \in O(n+1)$ so that

$$g = kk(g)k^{-1}ka(g)k^{-1}kn(g)k^{-1}$$

as above. Then $kk(g)k^{-1} \in O(n+1)$ and $ka(g)k^{-1}$ is uniformly bounded from I by a constant depending only on C_1 by Proposition 5.3. $kn(g)k^{-1} \in \mathbb{U}'$. Thus, we are done. □

A subset of a Lie group is of *polynomial growth* if the volume of the ball $B_R(\mathrm{I})$ radius R is less than or equal to a polynomial of R. As usual, the metric is given by the standard positive definite left-invariant bilinear form that is invariant under the conjugations by the compact group $O(n+1)$.

Lemma 5.19. $\mathbb{U}'$ *is of polynomial growth in terms of the distance from* I.

Proof. Clearly $\mathbb{U}$ is of polynomial growth since $\mathbb{U}$ is nilpotent as Gromov [33] showed. Given $g \in O(n+1)$, the distance between gug^{-1} and u for $u \in \mathbb{U}'$ is proportional to a constant multiplied by $\mathbf{d}(u, \mathrm{I})$: Choose $u \in \mathbb{U}'$ which is nilpotent. We can write $u(s) = \exp(s\vec{u})$ where $\vec{u}$ is a nilpotent matrix of unit norm. $g(t) := \exp(t\vec{x})$ for $\vec{x}$ in the Lie algebra of $O(n+1)$ of unit norm. For a family of $g(t) \in O(n+1)$, we define

$$u(t,s) = g(t)u(s)g(t)^{-1} = \exp(sAd_{g(t)}\vec{u}). \tag{48}$$

We compute

$$u(t,s)^{-1}\frac{du(t,s)}{t} := u(t,s)^{-1}(\vec{x}u(t,s) - u(t,s)\vec{x}) = (Ad_{u(t,s)^{-1}} - \mathrm{I})(\vec{x}).$$

Since $u(t,s)$ is nilpotent, $Ad_{u(t,s)^{-1}} - \mathrm{I}$ is a polynomial of variables t, s. The norm of $du(t,s)/dt$ is bounded above by a polynomial in s and t. The conjugation orbits of $O(n+1)$ in $\mathbf{Aut}(\mathbb{S}^n)$ are compact. Also, the conjugation by $O(n+1)$ preserves the distances of elements from I since the left-invariant metric μ is preserved by conjugation at I and geodesics from I go to geodesics from I of same μ-lengths under the conjugations by equation 48. Hence, we obtain a parametrization of $\mathbb{U}'$ by $\mathbb{U}$ and $O(n+1)$ where the volume of each orbit of $O(n+1)$ grows polynomially. Since $\mathbb{U}$ is of polynomial growth, we obtain that $\mathbb{U}'$ is of polynomial growth in terms of the distance from I. $\square$

We denote by $A'(g)$ the matrix obtained from $A(g)$ by adjoining the identity $(n - i_0) \times (n - i_0)$-matrix as below

$$A'(g) = \left(\begin{array}{c|c} \mathrm{I}_{n-i_0} & 0 \\ \hline 0 & A(g) \end{array}\right). \tag{49}$$

Lemma 5.20. *We can write g as a product of $K''(g)A'(g)U'(g)$ for a unipotent matrix $U'(g)$ where $K''(g)$ is of form*

$$K''(g) = \left(\begin{array}{c|c} K(g) & 0 \\ \hline 0 & K_2(g) \end{array}\right). \tag{50}$$

where $K_2(g)$ is an $(i_0+1) \times (i_0+1)$-orthogonal matrix. ($K(g)$ is not necessarily orthogonal.)

Proof. Consider

$$g' := \left(\begin{array}{c|c} K(g) & 0 \\ \hline 0 & \mathrm{I}_{i_0+1} \end{array}\right)^{-1} g. \tag{51}$$

Then g' can be decomposed as

$$g' = \left(\begin{array}{c|c} \mathrm{I}_{n-i_0} & 0 \\ \hline 0 & K_2(g) \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_{n-i_0} & 0 \\ \hline 0 & A(g) \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_{n-i_0} & 0 \\ \hline * & U(g) \end{array}\right). \tag{52}$$

□

5.5. **Closures of leaves.** Given a subgroup G of an algebraic Lie group, the *syndetic hull* $S(G)$ of G is a connected Lie group so that $S(G)/G$ is compact. (See Fried and Goldman [27] and D. Witte [60].)

The properly convex open set K has a Hilbert metric. We know that N_K acts isometrically on K. One can construct a Riemannian metric μ with bounded entries. We can assume that the derivatives of the entries of μ up to the m-th order are uniformly bounded above. Then $\{g^*\mu | g \in \mathbf{\Gamma}_E\}$ is an equicontinuous family up to any order. Thus the average of $g^*\mu$ for $g \in N_K$ is a C^m-Riemannian metric. This bestows us an N_K-invariant C^m-Riemannian metric μ_K on K.

The foliation on Ω_E given by fibers of π_K has leaves that are i_0-dimensional complete affine spaces. Since N_K is not discrete, there exists a component $N_{K,0}$ of the closure of N_K in $\mathbf{Aut}(K)$ and $N_{K,0}$ is a Lie group of dimension ≥ 1. We consider the frame bundle FK over K. A metric on each fiber of FK is induces from μ_K. Since the action of $N_{K,0}$ is isometric on FK with trivial stabilizers, we find that $N_{K,0}$ acts on a smooth orbit submanifold $\Sigma_{1,0}$ of FK transitively with trivial stabilizers. Then $\Sigma_{1,0}$ admits a smooth Riemannian metric $\mu_{0,1}$ invariant under $N_{K,0}$. (See Lemma 3.4.11 in [52].)

There exists a bundle $F\Omega_E$ from pulling back FK by the projection map. Since Γ_E acts isometrically on FK, the quotient space $F\Omega_E/\Gamma_E$ is a bundle $F\Sigma_E$ over Σ_E with a subbundle with compact fibers isomorphic to the orthogonal group of dimension $n - i_0$. Also, $F\Omega_E$ is foliated by i_0-dimensional affine spaces pulled-back from the i_0-dimensional leaves on the foliation Ω_E. Also, $F\Omega_E$ covers FE.

Each leaf l of $F\Omega_E$ goes to a point of FK. The point is acted upon by $N_{K,0}$ as an orbit of FK, whose inverse image becomes a smooth submanifold $\tilde{V}_l$ covering a compact submanifold V_l in $F\Sigma_E$ by the work of Molino [47]. Here l maps to a dense leaf in V_l.

Lemma 5.21. *Each leaf l is of polynomial growth. That is, each ball $B_R(x)$ of radius R has an area less than equal to $f(R)$ for a polynomial f where we are using arbitrary Riemannian metric on Ω_E and Σ_E so that the covering map $\Omega_E \to \Sigma_E$ is a local isometry.*

Proof. Let us choose a fundamental domain F of $F\Sigma_E$. Let $\hat{F}$ denote the image of F in FK. Then l is a union of $g_i(D_i)$ for the intersection D_i of a leaf with F where $g_i \in \Gamma_E$ for some index set I_l. We have that $D_i \subset D_i'$ where D_i' is an ϵ-neighborhood of D_i in the leaf. Then

$$\{g_i(D_i') | i \in I_l\}$$

cover l in a locally finite manner. The subset $G(l) := \{g_i \in \Gamma | i \in I_l\}$ is a discrete subset.

Choose an arbitrary point $d_i \in D_i$ for every $i \in I_I$. The set $\{g_i(d_i)|i \in I_I\}$ and I is quasi-isometric: there is a map from $G(I)$ to I given by $f_1 : g_i \mapsto g_i(d_i)$ and the multivalued map f_2 from I to $G(I)$ given by sending each point $x \in I$ to one of finitely many g_i such that $g_i(D_i') \ni x$. Both maps are quasi-isometries since these maps are restrictions of quasi-isometries $\mathbf{\Gamma}_E \to \Omega_E$ and $\Omega_E \to \mathbf{\Gamma}_E$ defined in an analogous manner.

The action of g_i in K is bounded since it moves some points of $\hat{F}$ to $\hat{F}$. Thus, $K(g_i)$ goes to a bounded subset of $\mathbf{Aut}(K)$. Hence

$$K(g_i) = \frac{1}{\det(K(g_i))^{1/(n-i_0)}}\hat{K}(g_i) \text{ where } \hat{K}(g_i) \in \mathsf{SL}_{\pm}(n-i_0, \mathbb{R}).$$

Let $\tilde{\lambda}_0(g_i)$ and $\tilde{\lambda}_n(g_i)$ denote the largest norm and the smallest norm of eigenvalues of $\hat{K}(g_i)$. These are bounded by two positive real numbers. The largest and the smallest eigenvalues of g_i equal

$$\frac{1}{\det(K(g_i))^{1/(n-i_0)}}\tilde{\lambda}_0(g_i) \text{ and } \frac{1}{\det(K(g_i))^{1/(n-i_0)}}\tilde{\lambda}_n(g_i)$$

Denote by $a_j(g_i), j = 1, \dots, i_0+1$, the norms of eigenvalues associated with $\mathbb{S}^{i_0}_\infty$. Since

$$\det(K(g_i))a_1(g_i)\dots a_{i_0+1}(g_i) = 1,$$

if $|\det(K(g_i))| \to 0$ or ∞, then the equation in Proposition 5.3 cannot hold. Therefore, we obtain

$$1/C < |\det(K(g_i))| < C$$

for a positive constant C. We deduce that the largest norm and the smallest norm of eigenvalues of g_i

$$\frac{1}{\det(K(g_i))^{1/(n-i_0)}}\tilde{\lambda}_0(g_i) \text{ and } \frac{1}{\det(K(g_i))^{1/(n-i_0)}}\tilde{\lambda}_n(g_i)$$

are bounded above and below by two positive numbers. Hence, $\lambda_0(g_i)$ and $\lambda_n(g_i)$ and the components of $a(g_i)$ are all bounded above and below by a fixed set of positive numbers.

By Corollary 5.18, $\{g_i\}$ is of bounded distance from $\mathbb{U}'$. Let $N_c(\mathbb{U}')$ be a c-neighborhood of $\mathbb{U}'$. Then

$$G(I) \subset N_c(\mathbb{U}').$$

By discreteness of $\mathbf{\Gamma}_E$, the set $\{g_i\}$ is discrete and there exists a lower bound

$$\{d(g_i, g_j)|g_i, g_j \in G(I), i \neq j\}.$$

Let $B_R(\mathrm{I})$ denote the ball in $\mathsf{SL}(n+1, \mathbb{R})$ of radius R. Then $B_R(\mathrm{I}) \cap N_c(\mathbb{U}')$ is of polynomial growth with respect to R. This implies that $G(I)$ is of polynomial growth with respect to the Cayley metric of $\mathbf{\Gamma}_E$.

Since the $\{g_i(D_i')|g_i \in G(I)\}$ of uniformly bounded balls cover I in a locally finite manner, I is of polynomial grow as well.

□

5.6. **The unipotency of N.** The foliation on Ω_E gives us a foliation of $F\Omega_E$. Let l be a leaf of $F\Omega_E$. Since l maps to a polynomial growth leaf in $F\Sigma_E$, by the work of Carrière [9], it follows that there exists a connected nilpotent Lie group $A_l \subset N_K$ acting on FK so that $A(l)$ form a submanifold $\tilde{V}$ of $F\Omega_E$ covering a compact submanifold V_l that is a leaf closure. Furthermore, A_l is the component of the closure of N_K the image of $\mathbf{\Gamma}_E$ in $\mathbf{Aut}(K)$.

Since A_l is in the product group, we can project to each Γ_i-factor or the central $\mathbb{R}^{l_0-1}$. One case is that Γ_j is not discrete in $\mathbf{Aut}(K_j)$ equal to $PO(n_j, 1)$ or $SO(n_j, 1)$ by [2]. The nilpotence implies that the image is a cusp group fixing a unique point in $\mathrm{bd}K_j$. Thus, the image is an abelian group since A_l is connected. In case, Γ_j is discrete, then a nilpotence implies that the image group fixes a unique pair of points in $\mathrm{bd}K_j$ and hence is abelian also. By connectedness, A_l maps to a trivial group here. Thus, the nilpotence implies that A_l is an abelian group.

Let N_l be exactly the subgroup of $\pi_1(V_l)$ fixing a leaf l in FK, for each closure V_l of a leaf l, the manifold V_l is compact and we have an exact sequence

$$1 \to N_l \to h(\pi_1(V_l)) \xrightarrow{\pi_K} A'_l \to 1$$

where A'_l is dense in A_l. Each leaf l' of Ω_E has a realization a subset in $\tilde{\mathcal{O}}$. Since N_l fixes points of K, we have $\lambda_0(g) = \lambda_1(g) = 1$ for $g \in N_l$. By Proposition 5.3, we have that N_l is virtually unipotent since the norms of eigenvalues equal 1 identically and N_l is discrete. (See the proof of Proposition 3.8 also.)

5.6.1. $US_{l,0}$ *is normalized by* $\mathbf{\Gamma}_E$. The leaf holonomy acts on $F\Omega_E/\mathcal{F}$ as an abelian killing field group without any fixed points. Hence, each leaf l is in $\tilde{V}_l$ with a constant dimension. Thus, $\mathcal{F}$ is a foliation with leaf closures of the identical dimensions. The leaf closures form another foliation $\bar{F}$ by Lemma 5.2 of Molino [48]. We let $FE/\bar{F}$ denote the space of closures of leaves has an orbifold structure where the projection $FE \to FE/\bar{F}$ is an orbifold morphism by Proposition 5.2 of [48]. Since Σ_E has a geometric structure induced from the transverse real projective structure, it follows that Σ_E is a very good orbifold. We may assume that E is an $n-1$-manifold and so is $F\Sigma_E$ since we need our results for finite index subgroups only. By Lemma 5.2 of [48], $F\Sigma_E/\bar{F}$ is the quotient space of $F\Omega_E/\mathcal{F}$ by the abelian killing field group. Thus, it admits a geometric structure induced from the real projective structure of $F\Omega_E/\mathcal{F}$. Thus, there exists a finite regular manifold-cover Σ^f of $F\Sigma_E/\bar{F}$ as in Chapter 13 of Thurston [51] (see [11] also.)

Hence, considering the fundamental groups, we obtain that there exists a regular finite cover $F\Sigma_E^f$ of $F\Sigma_E$ and a regular fibration

$$V_l \to F\Sigma_E^f \to \Sigma^f$$

where V_l is a generic fiber of FE^f for the induced foliation $\bar{F}^f$ isomorphic to a generic fiber of $F\Sigma_E$.

We obtain an exact sequence

$$\pi_1(V_l) \to \pi_1(F\Sigma_E^f) \xrightarrow{\pi'_K} \pi_1(\Sigma^f) \to 1$$

and the image $\pi_1(V_l)$ is a normal subgroup of $\pi_1(F\Sigma_E^f)$. We have a fibration

$$\widetilde{\mathsf{SO}}(n-i_0) \to F\Sigma_E^f \to \Sigma_E^f$$

where Σ_E^f is some regular finite cover of Σ_E and $\widetilde{\mathsf{SO}}(n-i_0)$ is a finite cover of $\mathsf{SO}(n-i_0)$. Thus, we also have an exact sequence

$$\pi_1(\widetilde{\mathsf{SO}}(n-i_0)) \to \pi_1(F\Sigma_E^f) \to \pi_1(\Sigma_E^f) \to 1.$$

Since $\pi_1(\Sigma_E^f)$ is a quotient group of $\pi_1(F\Sigma_E^f)$, it follows that the image of $\pi_1(V_l)$ is a normal subgroup of $\pi_1(\Sigma_E^f)$ for the generic l so that V_l in $F\Sigma_E^f$ to V_l in Σ_E^f is homeomorphic. We define $\mathbf{\Gamma}_l$ as the image $h(\pi_1(V_l))$. The above sequence tells us that $\mathbf{\Gamma}_l$ is a normal subgroup of $\mathbf{\Gamma}_E$.

Recall that A_l is abelian from Section 5.5 and $\pi_K(\mathbf{\Gamma}_E)$ normalizes A_l. Since A is a subgroup of the product of hyperbolic groups and abelian groups, A fixes a common set of fixed points on each factor K_i. It follows that the normalizer $A_E := \pi_K(\mathbf{\Gamma}_E)$ commutes with each element of A_l up to finite index also. We pass to the finite index subgroup as always and we assume that A_E is in the centralizer of A_l.

Let $\mathbf{\Gamma}_{l,l} := \mathbf{\Gamma}_l \cap N = N_l = N$ be the subgroup acting on each leaf of Ω_E. In the proof of Proposition 5.9 of [27], we take B to be the Zariski closure of $\mathbf{\Gamma}_{l,l} := \mathbf{\Gamma}_l \cap N$, which is virtually unipotent. B has only finitely many components and is an algebraic group. By Malcev's theorem, $B/\mathbf{\Gamma}_{l,l}$ is compact. Thus,

$$\mathbf{\Gamma}_E \cap N = \mathbf{\Gamma}_E \cap B = \mathbf{\Gamma}_l \cap B = \mathbf{\Gamma}_{l,l}.$$

We see that $A_l' = \mathbf{\Gamma}_l/(\mathbf{\Gamma}_l \cap B)$ embeds into its centralizer $A_E = \mathbf{\Gamma}_E/(\mathbf{\Gamma}_E \cap B)$.

Since $[A_l', A_l'] = \mathrm{I} \in \mathbf{Aut}(K)$, it follows that $[\mathbf{\Gamma}_l, \mathbf{\Gamma}_l] \subset \mathbf{\Gamma}_{l,l}$. Also, by the centralization conditions, for each element $g \in \mathbf{\Gamma}_E$ and $h \in \mathbf{\Gamma}_l$, we have $ghg^{-1} = hk'$ for $k' \in \mathbf{\Gamma}_l$.

We let $Z(\mathbf{\Gamma}_E)$ and $Z(\mathbf{\Gamma}_l)$ denote the Zariski closures of $\mathbf{\Gamma}_E$ and $\mathbf{\Gamma}_l$ respectively. We have an exact sequence

$$1 \to Z(\mathbf{\Gamma}_{l,l})(= B) \to Z(\mathbf{\Gamma}_E) \to \hat{A}_E \to 1.$$

Here, $\hat{A}_E$ is an algebraic group over $\mathbb{R}$ and is the Zariski closure of A_E since otherwise $Z(\mathbf{\Gamma}_E)$ has a smaller Zariski closure. Thus, $\hat{A}_E$ centralizes A_l' asA_E centralizes A_l'. (See Theorem 6.8 of [8].)

We also have

$$[Z(\mathbf{\Gamma}_l), Z(\mathbf{\Gamma}_l)] = Z([\mathbf{\Gamma}_l, \mathbf{\Gamma}_l]) \subset Z(\mathbf{\Gamma}_{l,l})$$

by Lemma 1.12 of [27]. We also have an exact sequence

$$1 \to Z(\mathbf{\Gamma}_{l,l})(= B) \to Z(\mathbf{\Gamma}_l) \to \hat{A}_l \to 1.$$

$\hat{A}_l$ is an abelian group since the commutator of the second group is in the kernel. Since both groups to the left have finitely many components, $\hat{A}_l$ has finitely many components. Thus, $\hat{A}_l$ is an abelian Lie group with finitely many components.

We have that $\hat{A}_l$ is a subset of $\hat{A}_E$. Hence, it follows that $\hat{A}_E$ is in the centralizer of $\hat{A}_l$. We have an inclusion

$$\frac{\mathbf{\Gamma}_l}{\mathbf{\Gamma}_l \cap B} \to \hat{A}_l.$$

We can find an abelian Lie group $\tilde{A}$ containing the image of this map with finitely many components so that $\tilde{A}$ mod $\tilde{A} \cap \mathbf{\Gamma}_l/(\mathbf{\Gamma}_l \cap B)$ is compact. Here $\tilde{A}$ may not be unique. The inverse image of A' under the quotient map

$$Z(\mathbf{\Gamma}_l) \to Z(\mathbf{\Gamma}_l)/B$$

has finitely many components since B and $\tilde{A}$ have finitely many components and contains $\mathbf{\Gamma}_l$. Let S_l denote the group. S_l is the syndetic hull of $\mathbf{\Gamma}_l$ as given as in Section 1.11 of [27], which means that $S_l/\mathbf{\Gamma}_l$ is compact and S_l has finitely many components. Note that the Zariski closure of S_l is same as the Zariski closure of $\mathbf{\Gamma}_l$.

Also, S_l is virtually solvable since B is unipotent and $\tilde{A}$ is abelian (see also Proposition 5.9 of David Witte [60]).

For each element g in A_E of $\mathbf{Aut}(K)$, g commutes with each element of $\tilde{A}$. Thus, for elements $g \in \mathbf{\Gamma}_E$, and $s \in S_l$, we have

$$\pi(gsg^{-1}) = \pi(g)\pi(s)\pi(g)^{-1} = \pi(s);$$

thus,

$$gsg^{-1} \in \pi^{-1}(\pi(s)) \in \pi^{-1}(\tilde{A}) = S_l.$$

Hence, S_l is normalized by $\mathbf{\Gamma}_E$. To summarize, we have by passing up to finite cover, we assume that $\mathbf{\Gamma}_E$ normalizes S_l.

Lemma 5.22. $h(\pi_1(V_l))$ *is virtually solvable and is contained in a virtually solvable Lie group* $S_l := S(h(\pi_1(V_l)))$ *with finitely many components, and* $S_l/h(\pi_1(V_l))$ *is compact. Furthermore, one can modify the strict end neighborhood* U *so that* S_l *acts on it. Also the Zariski closure of* $h(\pi_1(V_l))$ *is the same as that of* S_l*.*

Proof. Recall that Γ_l denotes $h(\pi_1(V_l))$. There exists a syndetic hull S_l so that S_l/Γ_l is compact by Section 5.6.1. By construction, Γ_l has the same Zariski closure as S_l.

Let F be a compact fundamental domain of S_l under the Γ_l-action. Then we have

$$\bigcap_{g \in S_l} g(U) = \bigcap_{g \in F} g(U).$$

Since F is compact, the latter set is still an end-neighborhood. □

Since S_l acts on U as shown in Lemma 5.22, we have a homomorphism $S_l \to Aut(K)$. We define by $S_{l,0}$ the kernel of this map. Then $S_{l,0}$ acts on each leaf of Ω_E. $S_{l,0}$ is the normal subgroup of S_l characterized by the condition that it acts on each leaf; that is, its element acts on each of the hemispheres of dimension $i_0 + 1$ with boundary $\mathbb{S}^{i_0}_\infty$. Since $gS_{l,0}g^{-1}$ for $g \in \mathbf{\Gamma}_E$ satisfies the same condition, it follows that $\mathbf{\Gamma}_E$ normalizes $S_{l,0}$.

Hence, we obtain

Proposition 5.23. $S_{l,0}$ *is a normalized by a finite index subgroup of* $\mathbf{\Gamma}_E$*.*

5.6.2. *The form of* $US_{l,0}$. From now on, we will let S_l to denote the only the identity component of itself for simplicity. This will be sufficient for our purpose of getting a cusp group normalized by $\mathbf{\Gamma}_E$.

Let US_l denote the unipotent radical of the Zariski closure of S_l, which is a solvable Lie group. Also, $US_{l,0}$ denote the unipotent radical of the Zariski closure of $S_{l,0}$. Then we let $\mathbf{\Gamma}_l$ be the intersection of $\mathbf{\Gamma}_l$ with S_l so that it is solvable now.

Proposition 5.24. *Let l be a generic fiber so that A_l acts with trivial stabilizer.*

- *S_l acts on $\tilde{V}_l$ and on Ω_E and ∂U properly and transitively and acts as isometries on these spaces with respect to Riemannian metrics.*
- *$S_{l,0}$ acts transitively on each leaf l with a compact stabilizer and acts on an i_0-dimensional ellipsoid passing $\mathbf{v}_E$ with an invariant Euclidean metric.*
- *$S_{l,0}$ is an i_0-dimensional cusp group and the unipotent radical $US_{l,0}$ is an i_0-dimensional abelian group equal to $S_{l,0}$.*
- *$US_{l,0}$ is normalized by $\mathbf{\Gamma}_E$ also.*

Proof. A stabilizer $S_{l,x}$ of each point $x \in \tilde{V}_l$ for S_l is compact: let F be the fundamental domain of S_l with Γ_l action. let F' be the image $F(x) := \{g(x)|g \in F\}$ in $\tilde{V}_l$. This is a compact set. $\Gamma_{l,F'}$ is defined as the set of $g \in \Gamma_l$ so that $g(F(x)) \cap F(x) \neq \emptyset$. Then $\Gamma_{l,F'}$ is finite by the properness of the action of Γ_l. Since an element of $S_{l,x}$ is a product of an element g' of Γ_l and $f \in F$, and $g'f(x) = x$, it follows that $g'F(x) \cap F(x) \neq \emptyset$ and $g' \in \Gamma_{l,F}$. Hence $S_{l,x} \subset \Gamma_{l,F'}F$ and $S_{l,x}$ is compact. Similarly, we see that S_l acts properly on Ω_E. Since ∂U is in one-to-one correspondence with Ω_E, we see that S_l acts on ∂U properly. Hence, these spaces have compact stabilizer with respect to S_l. The invariant metric follows from the compact stabilizer conditions.

Since the Lie group S_l acts with compact stabilizer, it follows that it preserves a Riemannian metric on Ω_E. Hence, the action is proper and the orbit is closed. Since $\tilde{V}_l/\Gamma_l$ is compact, $\tilde{V}_l/S_l$ is compact also.

Since l is a generic leaf, $\tilde{V}_l$ is a fiber bundle over A_l with fibers homeomorphic to cells. Thus $\tilde{V}_l$ is homotopy equivalent to a real abelian Lie group A_l, and to a torus T^{j_0} for some j_0. Also, there exists an exact sequence

$$1 \to B \to S_l \to \tilde{A} \to 1$$

giving us a fiber-bundle also. Since B is a finite extension of a unipotent group, it follows that S_l is homotopy equivalent to a finite cover of $\tilde{A}$. There is a surjective map $\tilde{A} \to A_l$ since $\mathbf{\Gamma}_l$ maps onto A'_l a dense subset. Thus, S_l is homotopy equivalent to T^{j_1} for $j_1 \geq j_0$.

Recall that $\mathbf{\Gamma}_l$ is solvable from above. By the last part of Section 1.8 of [27] where we can replace H there with S_l and $\mathbb{R}^n$ with $\tilde{V}_l$ and Γ with the solvable subgroup $\mathbf{\Gamma}_l$, we obtain the results for $\mathbf{\Gamma}_l$:

First, $\mathbf{\Gamma}_l$ is solvable and discrete, and hence is polycyclic and S_l has the same Zariski closure as $\mathbf{\Gamma}_l$. Take a finite index subgroup so that $\mathbf{\Gamma}_i$ is now polycyclic. We work on the projection of $\tilde{V}_l$ on Ω_E, a convex but not properly convex open domain. The proof identical with that of Lemma 1.9 of [27] shows that the unipotent radical US_l of $Z(S_l)$ acts freely on $\tilde{V}_l$. Being unipotent, US_l is simply connected. The orbit $US_l(x)$ for

$x \in \Omega_E$ is simply connected and invariant under $Z(\mathbf{\Gamma}_l)$. $US_l(x)/\mathbf{\Gamma}_l$ is a $K(\mathbf{\Gamma}_l, 1)$-space. Thus, $\mathsf{rank}\mathbf{\Gamma}_l = cd\mathbf{\Gamma}_l \leq \dim US_l$. By Lemma 4.3.6 of [49], $\dim US_l \leq \dim S_l$ and by Lemma 1.6 (iv), we have $\dim S_l \leq \mathsf{rank}\mathbf{\Gamma}_l$. Thus, $\mathsf{rank}\mathbf{\Gamma}_l = \dim S_l$. We now show S_l acts freely on Ω_E. We have a fibration sequence

$$\mathbf{\Gamma}_l \rightarrow S_l \rightarrow S_l/\mathbf{\Gamma}_l$$

and an exact sequence

$$\pi_1(S_l) \rightarrow \pi_1(S_l/\mathbf{\Gamma}_l) \rightarrow \mathbf{\Gamma}_l,$$

and hence $\mathsf{rank}\pi_1(S_l) + \mathsf{rank}\mathbf{\Gamma}_l = \mathsf{rank}\pi_1(S_l/\mathbf{\Gamma}_l) = \dim S_l$ since $S_l/\mathbf{\Gamma}_l$ is a compact manifold and by construction of S_l. (The quotient S_l/B is abelian where the analogous statement clearly holds by construction for the quotient by $\mathbf{\Gamma}_l/(\mathbf{\Gamma}_l \cap B)$ and since B is simply connected, this holds for $B/(B \cap \mathbf{\Gamma}_l)$) Since $\mathsf{rank}\mathbf{\Gamma}_l \geq \dim S_l$, we have $\mathsf{rank}\pi_1(S_l) = 0$. This means that $\pi_1(S_l)$ is finite. Being solvable, it is trivial. Thus, S_l is simply connected.

Since S_l acts with trivial stabilizers on Ω_E, it acts so on $\tilde{V}_l$. We showed that S_l acts freely on $\tilde{V}_l$. (We followed Section 1.8 of [27] faithfully here.)

We have a fibration

$$S_l \rightarrow \tilde{V}_l \rightarrow \tilde{V}_l/S_l.$$

Since $\tilde{V}_l/\Gamma_l$ is compact, so is $\tilde{V}_l/S_l$. $\tilde{V}_l/S_l$ is also a manifold. Let $j_2 := \dim \tilde{V}_l/S_l$. Then by the Leray-Hirsch theorem, the real homology of $\tilde{V}_l$ is isomorphic to the tensor product of those of S_l and $\tilde{V}_l/S_l$. The top homology dimension of $\tilde{V}_l$ is j_0 and that of S_l is j_1 and that of $\dim V_l/S_l$ is j_2. Since $j_0 \leq j_1$ and $j_0 = j_1 + j_2$. It follows that $j_2 = 0$ and V_l/S_l is a singleton. This means that S_l acts simply transitively on $\tilde{V}_l$.

Hence, $S_{l,0}$ acts simply transitively on each l; $S_{l,0}$ is diffeomorphic to a leaf l and hence is connected and is a solvable Lie group.

Since the subset $U_l := U \cap H_l^{i_0+1}$ of U corresponding to l is a strictly convex set containing v_E, we have $S_{l,0}$ acting simply transitively on ∂U_l. As before in the proof of Theorem 3.9 using the results of [25], we have $S_{l,0}$ acts on an i_0-dimensional ellipsoid that has to equal ∂U_l. Since one can identify each leaf with an affine space $S_{l,0}$ is isomorphic to an affine isometry group acting simply transitively on an affine space $\mathbb{R}^i$. Let $\mathcal{H}_{\mathsf{v}_E}$ denote the cusp group acting on the ellipsoid. An elementary argument using the cocompact subgroup simultaneously in both group shows that $S_{l,0}$ and $\mathcal{H}_{\mathsf{v}_E}$ are identical.

This shows also that $S_{l,0}$ is nilpotent and we have $US_{l,0} = S_{l,0}$ also. Finally, this implies that $US_{l,0}$ is an i_0-dimensional abelian Lie group. Since $S_{l,0}$ is normalized by $\mathbf{\Gamma}_E$ by Proposition 5.23, so is its unipotent radical $US_{l,0}$.

□

5.7. **The forms of Γ_E.**

5.8. **Existence of splitting.**

5.8.1. *Matrix form.* We can parametrize $US_{l,0}$ by $\mathcal{N}(\vec{v})$ for $\vec{v} \in \mathbb{R}^{i_0}$ by Proposition 5.24. As above by Lemma 5.7 and 5.10, we have that the matrices are of form.

$$\mathcal{N}(\vec{v}) = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}) & ||\vec{v}||^2/2 & \vec{v} & 1 \end{array}\right), \tag{53}$$

$$g = \left(\begin{array}{c|c|c|c} S(g) & 0 & 0 & 0 \\ \hline 0 & a_1(g) & 0 & 0 \\ \hline C_1(g) & a_4(g) & a_5(g)O_5(g) & a_6(g) \\ \hline c_2(g) & a_7(g) & a_8(g) & a_9(g) \end{array}\right) \tag{54}$$

where $g \in \mathbf{\Gamma}_E$. We denote by μ_g the $a_5(g)/a_1(g) = a_9(g)/a_5(g)$. Since S_l is in $Z(\mathbf{\Gamma}_l)$ and the orthogonality of normalized $A_5(g)$ is an algebraic condition, it follows that the above form also holds for $g \in S_l$.

Let N''' denote the group generated by $\mathbf{\Gamma}_E$ and S_l.

First, $N_K/N_K \cap A_l$ is discrete since A_l is the identity component of the closure of N_K in $\mathbf{Aut}(K)$. We conclude that N''' is closed: Suppose that we have $\{g_i\} \to g$ for elements $g_i, g \in N'''$. Then $\{\pi_K(g_i)\} \to \pi_K(g)$ and this implies that $\pi_K(g_i) \in \pi_K(g)N_K$ for $i > I_0$ for some I_0 by discreteness. Thus, $g^{-1}g_i \in \Gamma_l \subset S_l$ for $i > I_0$ and hence $g \in g_i S_l$ for all $i > I_0$. Hence, $g \in N''''$ and we obtain the closedness.

Let N'_K denote the group generated by A_l and N_K which is the closure of N_K. We have an exact sequence

$$1 \to N'''_l \to N''' \to N'_K \to 1$$

where N'_K acts on a properly convex domain K in $\mathbb{S}^{n-i_0-1}$ and we have $US_{l,0} \subset N'''_1$. Since S_l and $\mathbf{\Gamma}_E$ normalize $US_{l,0}$, so does N'''. Since N''' contains $\mathbf{\Gamma}_E$ as a matrix group, and μ and O_5 are obtained by taking entries of the matrices, it follows that μ and O_5 extend to N''' as homomorphisms. Since $\mu|N'''_l = 1$ and $O_5|US_{l,0} = \mathrm{I}$ by the matrix forms, it follows that there are homomorphisms

$$\mu : N''' \to \mathbb{R}^+, O_5 : N''' \to O(i_0), \text{ and } \mu O_5 : N''' \to \mathbb{R} \times O(i_0).$$

Proposition 5.25. *We have $\mu_g = 1$ for every $g \in N'''$.*

Proof. First suppose that $\mathbf{\Gamma}_{l,l}$ is trivial. Then N is trivial and $\mathbf{\Gamma}_E \to \mathbf{Aut}(K)$ is injective. The image N_K centralizes the image of $\mathbf{\Gamma}_l$ in $\mathbf{Aut}(K)$. This implies that $\mathbf{\Gamma}_E$ centralizes $\mathbf{\Gamma}_l$ also since $N = \{\mathrm{I}\}$. Here, $\mathbf{\Gamma}_l$ is abelian and then we can choose a syndetic hull S_l where $S_{l,0}$ is not trivial. $\mathbf{\Gamma}_E$ centralizes the Zariski closure $Z(\mathbf{\Gamma}_l)$ of $\mathbf{\Gamma}_l$. Since $S_l \subset Z(\mathbf{\Gamma}_l)$, it follows that $\mathbf{\Gamma}_E$ centralizes S_l and hence $US_{l,0}$. This implies $\mu_g = 1$ of course since $M_g = \mathrm{I}$ for all $g \in N'''$.

Suppose that $\mathbf{\Gamma}_{l,l}$ is finite. Then $\mathbf{\Gamma}_E$ acts as a group of finite automorphisms of Γ_l since a finite index subgroup of $\mathbf{\Gamma}_E$ centralizes S_l. This implies $\mu_g = 1$ for all $g \in \mathbf{\Gamma}_E$. Clearly, this is true for the Zariski closure and hence for N''' as well.

Suppose that $\Gamma_{l,l}$ is infinite. For some $h \in \Gamma_{l,l} - \{\mathrm{I}\}$, we have $h = kN(\vec{v}_h)$ for $k \in O(i_0)$. If $\mu_g \neq 1$ for $g \in \mathbf{\Gamma}_E$, we may assume without loss of generality that $\mu_g < 1$. We obtain that

$$g^n kN(\vec{v}_h) g^{-n} = g^n k g^{-n} N(\vec{v}_h \mu_g^n O_g^{5,-1}).$$

We can choose a subsequence that converges to an elliptic element k since the off-diagonal elements of $N(\vec{v})$ are linear functions of $\vec{v}$. Thus, $\Gamma_{l,l}$ contains elements arbitrarily close to k This contradicts the discreteness of Γ_l, and hence, $\mu_g = 1$.

Since $US_{l,0}$ is abelian, it follows that the conjugation by $g \in US_{l,0}$ gives us $\mu_g = 1$. Since N''' is generated by $\mathbf{\Gamma}_E$ and $US_{l,0}$, the result is clear.

□

We let $G = N'''$ above.

Proposition 5.26. *Let $\mathcal{O}$ be a properly convex real projective orbifold with radial ends. Let E be an end of $\mathcal{O}$ with an end neighborhood U and the end vertex $\mathbf{v}_E$. Assume that $M_g = O_5(g)^{-1}$ for all $g \in \mathbf{\Gamma}_E$. Assume also that $\mathbf{\Gamma}_E$ satisfies the weak uniform middle-eigenvalue conditions and $\mathbf{\Gamma}_E$ is in a closed Lie group G with the compact quotient $G/\mathbf{\Gamma}_E$ where $\pi_K(G)$ is a Lie group acting on K cocompactly and G contains S_l with kernel containing $\mathcal{N} = US_{l,0}$. Then there is a projective embedding of K'' in the closure of $\partial\tilde{\mathcal{O}}$ invariant under Γ, and one can find a coordinate system so that for every $\mathcal{N}(\vec{v})$ is written so that*

- $C_1(\vec{v}) = 0$ *and* $c_2(\vec{v}) = 0$ *for every* $\vec{v} \in \mathbb{R}^{i_0}$ *and*
- $C_1(g) = 0$ *and* $c_2(g) = 0$ *for* $g \in \mathbf{\Gamma}_E - N$.

And our end E is a join or quasi-join type of cone over a totally geodesic domain K'' and a cusp end with a common end vertex $\mathbf{v}_E$.

Proof. Since $\mathbf{\Gamma}_E$ normalizes $US_{l,0}$, Lemmas 5.7 and 5.10 apply to this situation. Since K/N_K is compact, K has a compact set F which every orbit meets. K is foliated by open lines from a point $k \in \mathrm{Cl}(K)$ to points of open convex domain K'' of dimension $n - i_0 - 1$. That is, K is the interior of the join $k * \mathrm{Cl}(K'')$. Call these k-radial lines. Take such a line l and a sequence of points $k_m \to k_\infty \in K''$ as $m \to \infty$. Hence, there exists a sequence $\{\gamma_m\}$ of elements of Γ so that $\gamma_m(k_m) \in F$ and $\gamma_m(l)$ is a line passing F so that $\gamma_m(\partial_1 l) \to k_\infty$ for the endpoint $\partial_1 l$ of l in K''. Since K'' is properly convex, this implies $\{\gamma_m | K''\}$ is a bounded sequence of transformations and hence γ_m is of form:

$$\left(\begin{array}{c|c|c|c} \delta_m O_m & 0 & 0 & 0 \\ \hline 0 & \lambda_m & 0 & 0 \\ \hline C_{1,m} & \vec{v}_m^T & \lambda_m O_5(\gamma_m) & 0 \\ \hline c_{2,m} & a_7(\gamma_m) & \vec{v}_m O_5(\gamma_m) & \lambda_m \end{array}\right) \quad (55)$$

where O_m is a bounded sequence of matrices in $\mathbf{Aut}(K'')$ in $\mathsf{SL}(n - i_0 - 1, \mathbb{R})$ since the set of projective automorphisms of K'' moving points only bounded distances in $d_{K''}$

is bounded, and we have $\mu_g = 1$ identically by Proposition and hence Lemma 5.11 applies. Note that $\delta_m^{n-i_0-1}\lambda_m^{i_0+2} = 1$, and $\delta_m/\lambda_m \to 0$ as $\gamma_m|I$ pushes the points toward the vertex k of K.

We assume by choosing subsequences so that $\{O_m\}$ converges to $O_\infty \in \mathbf{Aut}(K'')$ and $\delta_m \to 0$ and $\lambda_m \to \infty$ for the determinant ± 1 representation of γ_m.

Note that γ_m acts on a (i_0+1)-dimensional hemisphere $H_1^{i_0+1}$ with boundary $\mathbb{S}^i_\infty$ corresponding to the vertex k of K. This also corresponds to the lower-right $(i_0+1) \times (i_0+1)$-submatrix. γ_m restricts to a cusp group with norms of eigenvalue equal to 1 and acts as a Euclidean isometry on an affine space A_1^i as space of complete lines in $H_1^{i_0+1}$ ending in $\mathbf{v}_E$.

Recall $\mathcal{N} = US_{l,0}$. Suppose that there exists a point x' of $\mathrm{Cl}(\tilde{\mathcal{O}})$ in the interior of $H_1^{i_0+1}$, i.e., $H_1^{i_0+1,o} \cap \mathrm{Cl}(\tilde{\mathcal{O}}) \neq \emptyset$. Since we know that $US_{l,0}$ acts on $\mathrm{Cl}(\tilde{\mathcal{O}})$ and $H_1^{i_0+1,o}$, it follows that $US_{l,0}$ acts on $H_1^{i_0+1,o} \cap \mathrm{Cl}(\tilde{\mathcal{O}})$. The orbit of x' under $US_{l,0}$ in $H_1^{i_0+1}$ is an ellipsoid E_2 and is properly convex.

The only other possibility is $H_1^{i_0+1} \cap \mathrm{Cl}(\tilde{\mathcal{O}}) = \{\mathbf{v}_E\}$ since otherwise $\mathbb{S}^{i_0}_\infty \cap \mathrm{Cl}(\tilde{\mathcal{O}}) \ni y$ for $y \neq \mathbf{v}_E$. (As in the proof of Lemma 5.8, this gives us a contradiction.)

Let S_m denote the subspace spanned by the Jordan-block subspaces corresponding to eigenvalues with norms strictly smaller than λ_m. Then we obtain $\dim S_m = \dim K''$, and there is a projection $S_m \to K'' \subset \mathbb{S}^{n-i_0-2}$ given by $\mathbb{S}^n - \mathbb{S}^{i_0+1}_\infty$ where $\mathbb{S}^{i_0+1}$ is the great sphere containing $\mathbb{S}^{i_0}_\infty$ and $i_0 + 1$-dimensional hemisphere with this boundary corresponding to k'. Let K''_m denote the corresponding properly convex subset of S_m to K''.

We introduce a coordinate on $\mathbb{S}^n$ respecting the matrix form of equation 55. Thus, given a vector $\vec{a} := (\vec{a}_1, a_2, \vec{a}_3, a_4)$ where $\vec{a}_1$ is a $n - i_0 - 1$-vector and $\vec{a}_3$ is i_0-vector and a_4 is the coordinate in the direction of $\mathbf{v}_E$ At least one point x_{m_0} of K''_{m_0} is in $\mathrm{bd}\tilde{\mathcal{O}}$ since we can apply $\gamma^i_{m_0}$ to a point of U as $i \to \infty$. Since $x_{m_0} \neq \mathbf{v}_E, \mathbf{v}_{E-}$, it follows that x_{m_0} has coordinates $\vec{a}_1 \neq 0$. Recall that

$$\mathcal{N}(\vec{v}) = \left(\begin{array}{c|c|c|c} \mathrm{I}_{n-i_0-1} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ \hline 0 & \vec{v}^T & \mathrm{I}_{i_0} & 0 \\ \hline c_2(\vec{v}) & \frac{||\vec{v}||^2}{2} & \vec{v} & 1 \end{array}\right). \tag{56}$$

In other words, on K''_{m_0}, $a_2 = 0$ and $\vec{a}_3 = 0$. We note that $c_2(\vec{v})$ is linear in $\vec{v}$ in this situation.

Now we choose the point x_{m_0} in K''_{m_0} and represent it as the vector $\vec{a} = (\vec{a}_1, 0, \vec{0}, a_4)$. We note that

$$\mathcal{N}(n\vec{v})\vec{a} = (\vec{a}_1, 0, \vec{0}, nc_2(\vec{v}) \cdot \vec{a}_1 + a_4) \text{ for } n \in \mathbb{Z}. \tag{57}$$

Hence, one can use $n \to \infty$ and $n \to -\infty$, we obtain two antipodal points, which contradicts the proper-convexity. Therefore, it must be that $c_2(\vec{v}) = 0$ in this coordinate as well.

By Equation 31, $C_1(g) = 0$ for all $g \in \mathbf{\Gamma}_E$ since $c_2(\vec{v}) = 0$ for all $\vec{v} \in \mathbb{R}^i$.

Let $\mathbb{S}^{n-i_0-1}_{m_0}$ denote the minimal subspace in $\mathbb{S}^n$ containing K''_{m_0} and $\mathbf{v}_E, \mathbf{v}_{E-}$. We define the tube $B(K''_{m_0})$ that is the union of segments passing K''_{m_0} with endpoints $\mathbf{v}_E, \mathbf{v}_{E-}$. Since $C_1(g) = 0$, $g \in \mathbf{\Gamma}_E$ acts on $\mathbb{S}^{n-i_0-1}_{m_0}$. Since g acts on K', it follows that g acts on $B(K''_{m_0})$, the subset of $\mathbb{S}^{n-i_0-1}_{m_0}$ corresponding to K' under the projection π_K. (This follows really since $\mathbb{S}^{n-i_0-1}_{m_0}$ is the unique maximal set of fixed points of $\mathcal{N} = \{\mathcal{N}(\vec{v}) | \vec{v} \in \mathbb{R}^i\}$ and the normal property.)

Since $\mathbf{\Gamma}_E$ acting on $B(K''_{m_0})$ thus satisfies the weak uniform middle-eigenvalue condition, Theorem 3.5 implies that Γ_E acts on a compact set K' distanced from $\mathbf{v}_E$ and $\mathbf{v}_{E-}$.

Since γ_m acts on K' as well, and we have $\delta_m/\lambda_m \to 0$ and $O_m \to O_\infty \in \mathbf{Aut}(K'')$. Suppose that $K' \neq K''_{m_0}$. Then $\gamma^i_{m_0}(K')$ has to become strictly closer to K''_{m_0} than K' for i sufficiently large by the matrix form of equation 55 as γ_{m_0} acts on $B(K''_0)$ with only invariant subspaces K''_m or one in $\{\mathbf{v}_E, \mathbf{v}_{E-}\}$. However, we have $\gamma^i_{m_0}(K') = K'$, an invariant set. Hence, we have $K' = K''_{m_0}$ for sufficiently large m_0. (Thus, K''_{m_0} is defined independently of m.)

It also follows that K' is in a totally geodesic hypersurface H' of dimension $n - i_0 - 1$ and $K' = B(K''_{m_0}) \cap H'$ since K' meets every complete segment in $B(K''_{m_0})$ with vertices $\mathbf{v}_E$ and $\mathbf{v}_{E-}$ and we can show $\gamma_m(K')$ goes into K' properly if K' has a nonempty interior.

Since g acts on K''_{m_0} and $\mathbf{v}_E, \mathbf{v}_{E-}$, it follows that $C_1(g) = 0$ and $c_2(g) = 0$ for all $g \in \mathbf{\Gamma}_E$ under this system of coordinates.

If $\alpha_7(g) = 0$ for all $g \in \mathbf{\Gamma}_E$, $\mathcal{N}$ acts on a horosphere H in $\mathbb{S}^{i_0+1}_\infty$ with the vertex $\mathbf{v}_E$. The join of H and K' form the join neighborhood that we wished to obtain by Proposition 5.14. If $\alpha_7(g) > 0$ for $g \in \mathbf{\Gamma}_{E,+}$, we obtained a quasi-joined end again by Proposition 5.14. $\square$

5.9. Non-existence of split joined cases as well: The proof of Theorem 0.1.

Proof. Earlier, in the proof of Proposition 5.13, we obtained γ_m of form:

$$\left(\begin{array}{c|c|c|c} \delta_m O_m & 0 & 0 & 0 \\ \hline 0 & \lambda_m & 0 & 0 \\ \hline 0 & \lambda_m \vec{v}_m^T & \lambda_m O_5(\gamma_m) & 0 \\ \hline 0 & \lambda_m \left(\alpha_7(\gamma_m) + \frac{||\vec{v}_m||^2}{2}\right) & \lambda_m \vec{v}_m & \lambda_m \end{array}\right) \tag{58}$$

where we showed $C_{1,m} = 0$ and $c_{2,m} = 0$. We used here the argument of the proof of Theorem 5.16.

By Proposition 5.8.1, we know $\mu_g = 1$ for all $g \in \Gamma_E$. In the proofs of Proposition 5.26 we change the sequence of element $\gamma_m \in \Gamma$ of form

$$\begin{pmatrix} \begin{array}{c|c|c|c} \delta_m O_m & 0 & 0 & 0 \\ \hline 0 & \lambda_m & 0 & 0 \\ \hline 0 & 0 & \lambda_m O_5(\gamma_m) & 0 \\ \hline 0 & \lambda_m \alpha_7(g) & 0 & \lambda_m \end{array} \end{pmatrix} \tag{59}$$

by multiplying by an element of $\mathcal{N}$ as we showed later $C_1(\gamma_m) = 0, c_2(\gamma_m) = 0$.

In the case when N_K is discrete, Theorem 5.16 proves our theorem.

When N_K is indiscrete case, we use

$$1 \to N_l''' \to N''' \to N_K \to 1.$$

We note that $\Gamma_E \subset N'''$ here.

We just need to show that the joined end does not occur. Thus we assume that $\alpha_7(g) = 0$ for all $g \in \mathbf{\Gamma}_E$. The arguments for the quasi-joined end follow from Proposition 5.14.

Note that the lower-right $(i_0+2) \times (i_0+2)$-matrix of the above matrix must act on the horosphere H. $\mathcal{N}$ also act transitively on H. Hence, for any such matrix we can find an element of $\mathcal{N}$ so that the product is in the orthogonal group acting on H.

Denote by $S(K')$ and $S(H)$ the subspaces determined by K' and H and containing them. $S(K')$ and $S(U)$ form a pair of complementary subspaces in $\mathbb{S}^n$.

We have the sequence γ_m acting on $K'_{\max}$ is uniformly bounded and γ_m acting on $H_{\max}$ in a uniformly bounded manner as $m \to \infty$. Let $H_{\max}$ denote $S(H) \cap \mathrm{Cl}(\tilde{\mathcal{O}})$ and $K'_{\max}$ the set $S(K') \cap \mathrm{Cl}(\tilde{\mathcal{O}})$. By Lemma 4.10 for $l = 2$ case, $\mathrm{Cl}(\tilde{\mathcal{O}})$ equals the convex hull of $H_{\max}$ and $K'_{\max}$.

Note that a group of parabolic automorphisms in Γ act on $\partial H_{\max} - \{\mathbf{v}_E\}$ cocompactly. The only $\mathbf{\Gamma}_E$-invariant subset of $\partial H_{\max}$ is $\mathbf{v}_E$ and its complement. The maximal join decomposition of $H_{\max}$ if it exists, have finitely many compact convex subsets and they are permuted by $\mathbf{\Gamma}_E$. The argument as in the last part of the proof of Lemma 4.9 gives us contradiction again to the assumption of the finite-essential-annulus condition.

The maximal join decomposition of $\mathrm{Cl}(H)$ then has $P(H)$ as a factor. Since $\pi_1(\mathcal{O})$ permutes the factors, it follows that $P(H_{\max}) = P(H)$ is $\pi_1(\mathcal{O})$-invariant. This implies that $\mathbf{\Gamma}$ is reducible. Hence the joined ends cannot occur.

If $\mathbf{\Gamma}_E$ satisfies the uniform middle-eigenvalue condition, and the associated end is not properly convex, then elements of $\mathbf{\Gamma}_E - \mathcal{N}$ do not satisfy this condition as γ_m above has $\lambda_1(\gamma_m) = \lambda_{\mathbf{v}_E}(g)$. Hence, we have only lens-type ends or cusp ends.

□

Appendix A. The affine action dual to the tubular action

Let Γ be an affine group acting on the affine space A^n with boundary ∂A^n in $\mathbb{S}^n$. Recall that Γ is asymptotically nice if there exists a properly convex invariant Γ-invariant domain U' with boundary in a properly convex domain $\Omega \subset \partial A^n$ and U' is in the intersection of all half-spaces H supporting U' at $\mathrm{bd}\Omega$.

Each element of g is of form

$$\begin{pmatrix} \frac{1}{\lambda_{\mathbf{v}_E}(g)^{1/n}}\hat{h}(g) & \vec{b}_g \\ \vec{0} & \lambda_{\mathbf{v}_E}(g) \end{pmatrix} \tag{60}$$

where $\vec{b}_g$ is $1 \times n$-vector and $\hat{h}(g)$ is an $n \times n$-matrix of determinant ± 1 and $\lambda_{\mathbf{v}_E}(g) > 0$ is a constant. In the affine coordinates, it is of form

$$x \mapsto \frac{1}{\lambda_{\mathbf{v}_E}(g)^{1+\frac{1}{n}}}\hat{h}(g)x + \frac{1}{\lambda_{\mathbf{v}_E}(g)}\vec{b}_g.$$

As above, if there exists a uniform constant $K > 0$ so that

$$K^{-1}\mathrm{length}(g) \leq \log \frac{\lambda_1(g)}{\lambda_{\mathbf{v}_E}(g)} \leq K\mathrm{length}(g), \quad g \in \mathbf{\Gamma}_E - \{\mathrm{I}\},$$

then Γ is said to satisfy the *uniform middle-eigenvalue condition*.

In this appendix, it is sufficient for us to prove when Γ is a hyperbolic group. However, we might be tempted to work out the case when Γ is a linear group with a trivial virtual center in the future.

Theorem A.1. *We assume that Γ is a hyperbolic group. Let Γ have a properly convex affine action on the affine space A^n acting on a properly convex domain $U \subset A^n$ with boundary in the convex domain $\mathrm{Cl}(\Omega)$ for a properly convex domain Ω in ∂A^n. Suppose that Ω/Γ is a closed $n-1$-dimensional orbifold and Γ satisfies the uniform middle-eigenvalue condition. Then Γ is asymptotically nice and the set of asymptotic hyperspaces at each boundary point of Ω is uniquely determined.*

It is fairly easy to show that this holds also for virtual products of hyperbolic and abelian groups as well. We omit the proof here. But it is contained in the proof of Proposition 3.7. In the case when the linear part of the affine maps have the determinant 1. Theorem 8.2.1 of Labourie [43] shows that such a domain U exists. In general, we think that the existence of the domain U can be obtained but the proof is much longer. (See Appendix of [19] in the special case that can be extended here.)

A.1. **Anosov flow.** We generalize the work of Goldman-Labourie-Margulis [32]: Assume as in the premise of Theorem A.1. Since Ω is properly convex, Ω has a Hilbert metric. Let $U\Omega$ denote the unit tangent bundle over Ω. This has a smooth structure as a quotient space of $T\Omega - O/\sim$ where O is the image of the zero-section and $\vec{v} \sim \vec{w}$ if $\vec{v}$ and $\vec{w}$ are over the same point of Ω and $\vec{v} = s\vec{w}$ for a real number $s > 0$. (Clearly $U\Omega$ itself is not a smooth submanifold of $T\Omega$ however.)

Assume Γ as above. We note that $\Sigma := \Omega/\Gamma$ is a properly convex convex real projective orbifold. Thus, $U\Sigma := U\Omega/\Gamma$ is a compact smooth orbifold again. There

exists a geodesic flow on $U\Omega/\Gamma$ that is Anosov and hence topologically mixing. Hence, the flow is nonwondering everywhere. (See [1].) Γ acts irreducibly on Ω and $\mathrm{bd}\Omega$ is C^1.

We form the product $U\Omega \times A^n$ that is an affine bundle over Ω. We take the quotient $U\Omega \times A^n$ by the diagonal action

$$g(x, \vec{u}) = (g(x), Dg\vec{u}) \text{ for } g \in \Gamma, x \in U\Omega, \vec{u} \in A^n.$$

We denote the quotient by $\mathbb{V}$ fibering over the smooth orbifold $U\Omega/\Gamma$ with fiber A^n.

Let V^n be the vector space associated with A^n. Then we can form $U\Omega \times V^n$ and take the quotient under the diagonal action:

$$g(x, \vec{u}) = (g(x), \mathcal{L} \circ Dg\vec{u}) \text{ for } g \in \Gamma, x \in U\Omega, \vec{u} \in V^n$$

where $\mathcal{L}$ is the homomorphism taking the linear part of g. We denote by $\mathbb{V}$ the fiber bundle over $U\Omega/\Gamma$ with fiber V^n.

Since $U\Omega \times A^n$ is a flat A^n-bundle over $U\Omega$, we have a flat connection ∇^A on the bundle $\mathbb{V}$ over $U\Sigma$ and a flat linear connection ∇^V on the bundle $\mathbb{V}$ over $U\Sigma$. The connections are simply ones induced by the trivial product structure.

We give a decomposition of $\mathbb{V}$ into three parts $\mathbb{V}_+, \mathbb{V}_0, \mathbb{V}_-$: For each vector $\vec{u} \in U\Omega$, we find the maximal geodesic l ending at two points $\partial_+ l, \partial_- l$. They correspond to the 1-dimensional vector subspaces V_+ and V_-. There exists unique supporting hypersphere H_+ and H_- in ∂A^n at each of $\partial_+ l$ and $\partial_- l$. We denote by $H_0 = H_+ \cap H_-$. It is a codimension 2 great sphere in ∂A^n and corresponds to a vector subspace V_0 of codimension two in $\mathbb{V}$. For each vector $\vec{u}$, we find the decomposition of V as $V_+ \oplus V_0 \oplus V_-$ and hence we can form the subbundles $\mathbb{V}_+, \mathbb{V}_0, \mathbb{V}_-$ where $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_0 \oplus \mathbb{V}_-$.

If $g \in \Gamma$ acts on l, then V_+ and V_- are eigenspaces of the largest norm $\lambda_1(g)$ of the eigenvalues and the smallest norm $\lambda_n(g)$ of the eigenvalues of the linear part of g equal to

$$\frac{1}{\lambda_{\mathbf{v}_E}(g)^{(n+1)/n}} \hat{h}(g).$$

Hence on V_+, g acts by expending by $\lambda_1(g)/\lambda_{\mathbf{v}_E}(g)$ and on V_-, g acts by contracting by $\lambda_n(g)/\lambda_{\mathbf{v}_E}(g)$.

There exists a flow $\Phi_t : U\Sigma \to U\Sigma$ for $t \in \mathbb{R}$ given by sending $\vec{v}$ to the unit tangent vector to at $\alpha(t)$ where α is a geodesic tangent to $\vec{v}$ with $\alpha(0)$ equal to the base point of $\vec{v}$.

Now we define a flow $\hat{\Phi}_t : \mathbb{V} \to \mathbb{V}$ by lifting the flow. We define a flow on $\tilde{\Phi}_t : \mathbb{V} \to \mathbb{V}$ by considering a unit speed geodesic flow line $\vec{l}$ in $U\Omega$ and and considering $\vec{l} \times E$ and acting trivially on the second factor as we go from $\vec{v}$ to $\Phi_t(\vec{v})$ (See remarks in the beginning of Section 3.3 and equations in Section 4.1 of [32].) Each flow line in $U\Sigma$ lifts to a flow line on $\mathbb{V}$ from every point in it.

We define a flow on $\tilde{\Phi}_t : \mathbb{V} \to \mathbb{V}$ by considering a unit speed geodesic flow line $\vec{l}$ in $U\Omega$ and and considering $\vec{l} \times V$ and acting trivially on the second factor as we go from $\vec{v}$ to $\Phi_t(\vec{v})$ for each t. (This generalizes the flow on [32].)

As in Section 4.4 of [32], $\mathbb{V} = \mathbb{V}_+ \oplus \mathbb{V}_0 \oplus \mathbb{V}_-$. By the uniform middle-eigenvalue condition, there exists a fiberwise Euclidean metric g on $\mathbb{V}$ with the following properties:

- the flat linear connection ∇_V is bounded with respect to g.

- hyperbolicity: There exists constants $C, k > 0$ so that

$$||\tilde{\Phi}_t(\vec{v})|| \geq \frac{1}{C}\exp(kt)||\vec{v}|| \text{ as } t \to -\infty \tag{61}$$

for $\vec{v} \in \mathbb{V}_+$ and

$$||\tilde{\Phi}_t(\vec{v})|| \leq C\exp(-kt)||\vec{v}|| \text{ as } t \to \infty \tag{62}$$

for $v \in \mathbb{V}_-$.

The construction of such metric g is given by choosing one for A^n and extending it on $\Omega \times A^n$ by choosing a cover of Ω/Γ by compact sets K_i and choosing the extension over $K_i \times A^n$. Then we use the partition of unity.

Proposition A.2 proves this property by taking C sufficiently large according to t_1, which is a standard technique.

A.1.1. *The proof of the Anosov property.* Note the $\mathbb{V}_+$ and $\mathbb{V}_-$ are C^0-bundles over $U\Sigma$ since for each geodesic the ends points correspond to the line bundle.

We can apply this to $\mathbb{V}_-$ and $\mathbb{V}_+$ by possibly reversing the direction of the flow. The Anosov property follows from the following proposition.

Proposition A.2. *Let S be a closed real projective orbifold with hyperbolic group. Then there exists a constant t_1 so that*

$$||\Phi_t(\mathbf{v})|| \leq \tilde{C}||\mathbf{v}||$$

for $t \geq t_1$ and a uniform $\tilde{C}$, $0 < \tilde{C} < 1$ for $\mathbf{v} \in \mathbb{V}_-$.

Proof. Let $\mathbb{V}_{-,1}$ denote the subset of $\mathbb{V}_-$ of unit length.

By the following Lemma A.3, we have that for given $\epsilon > 0$, for each vector $\mathbf{v}$ in $\mathbb{V}_-$, there exists T so that for $t > T$, $\Phi_t(\mathbf{v})$ is in ϵ-neighborhood $U_\epsilon(S_0)$ of S_0. We choose an open cover of the space $\mathbb{V}_{-,1}$ of unit vectors of $\mathbb{V}_-$ by open sets U_i with T_i so that $\Phi_t(U_i)$ is a subset of $U_\epsilon(S_0)$ for $t > T_i$. Since $\mathbb{V}_{-,1}$ is compact, we can find a finite cover $U_i, i \in I_1$ and $t_1 = \max\{T_i\}_{i \in I_1}$ that we desired. □

The line bundle $\mathbb{V}_-$ lifts to $\tilde{\mathbb{V}}_-$ where each unit vector $\mathbf{u}$ on Ω one associates the line $\mathbb{V}_{-,\mathbf{u}}$ corresponding to the end point $\mathrm{bd}\Omega$ of the geodesic tangent to it. Φ lifts to a parallel translation or constant flow of $\tilde{\mathbb{V}}_-$ fixing each vector $\mathbf{v}$.

Lemma A.3. *Let $C > 0$ be an arbitrary constant. For each point $\mathbf{v}$ of $\mathbb{V}_-$, there is no sequence t_i, $t_i \to \infty$ so that $||\Phi_{t_i}(\mathbf{v})|| \geq C$ for a constant $C > 0$.*

Proof. Suppose not. Then there exists a sequence $\{t_i\}$ where $\{t_i\} \to \infty$ so that we also have $\{\Phi_{t_i}(\mathbf{v})\} \to \mathbf{v}_\infty$ for $\mathbf{v}_\infty \in \mathbf{v}_-$ or $\{||\Phi_{t_i}(\mathbf{v})||\} \to \infty$ and we assume that the projection $y_i := \Pi(\Phi_{t_i}(\mathbf{v})) \to y_\infty \in U\Sigma$ where $\mathbf{v}_\infty$ lies over y_∞.

By construction, we recall that each y_i is in the flow line, say l. We can choose lifts $\tilde{y}_i$ on a geodesic l so that $\{\tilde{y}_i\}$ converges to a point $\tilde{y}_\infty$, an endpoint of l. Let z be the other endpoint of l.

Find a fundamental domain F of $U\Omega$, and find a deck transformation g_i so that $g_i(\tilde{y}_i) \in F$ and g_i acts on the line $\tilde{\mathbf{v}}_-$ by the linearization of the matrix of form of

equation 60: i.e., by

$$\mathcal{L}(g_i) := \frac{1}{\lambda_{\mathbf{v}_E}(g_i)^{1+\frac{1}{n}}}\hat{h}(g)|\mathbf{v}_{-,\tilde{y}_\infty}$$

where $\mathbf{v}_{-,\tilde{y}_\infty}$ is over $\tilde{y}_\infty$ mapping to another line L' over the point $g_i(y_\infty)$.

Since Γ is hyperbolic, the action of Γ is a convergence group (See [13] and [1].) We can choose g_i so that there exist two points a_∞ and r_∞ where $g_i|\mathrm{bd}\Omega - \{r_\infty\}$ has the property that every compact subset converges to $\{a_\infty\}$ and $g_i^{-1}|\mathrm{bd}\Omega - \{a_\infty\}$ has the property that the sequence restricted to every compact subset converges to $\{r_\infty\}$ uniformly. If $a_\infty \neq r_\infty$, then we choose an element $g \in \Gamma$ so that $g(a_\infty) \neq r_\infty$ and replace the sequence by $\{gg_i\}$ and replace F by $F \cup g(F)$.

We choose a subsequence of $\{g_i\}$ so that for the attracting fixed point $a_i \in \mathrm{Cl}(\Omega)$ and the repelling fixed point $r_i \in \mathrm{Cl}(\Omega)$ of each g_i, it follows that $\{a_i\}$ and $\{r_i\}$ are convergent. Then it is clear that $\{a_i\} \to a_\infty$ and $r_i \to r_\infty$.

We will use the standard metric on $\mathbb{R}^{n+1}$. We may assume without generality that $\{g_i(y_i)\} \to y'$ for $y' \in F$. Each $g_i(\overline{zy_\infty})$ passes F and we may assume without loss of generality that $\{g_i(\overline{zy_\infty})\}$ converges to a nontrivial line in Ω geometrically by choosing subsequences. Hence, $\{g_i(z)\} \to z_\infty$ for a point $z_\infty \in \mathrm{bd}\Omega$ and $\{g_i(y_\infty)\} \to y'$ where we must $z_\infty \neq y'$.

Since Γ acts as a convergence group, we see that g_i restrict to any compact subset of $\mathrm{bd}\Omega - \{y'\}$ converges to the constant map with value $\{a_\infty\}$. Hence, we obtain $a_\infty = z_\infty$.

Also, g_i has an invariant great sphere $\mathbb{S}_i^{n-1}$ containing the attracting fixed point a_i and r_i is uniformly bounded at a distance from $\mathbb{S}_i^{n-1}$ as $\{r_i\} \to y'$. Since $g_i(y_\infty)$ is a point of $\mathrm{bd}\Omega$ and is uniformly bounded away from a_i. Therefore, $g_i(y_\infty)$ is also uniformly bounded away from the tangent sphere $\mathbb{S}_i^{n-1}$ at a_i.

Let $\vec{v}_i$ denote the unit vector in direction of y_i. Then $\vec{v}_i$ has the component $\vec{v}_i^P$ parallel to r_i and the component $\vec{v}_i^S$ in the direction of $\mathbb{S}_i^{n-1}$ where $\vec{v}_i = \vec{v}_i^P + \vec{v}_i^S$. g_i acts by preserving the directions of $\mathbb{S}_i^{n-1}$ and r_i. Since $\{g_i(y_i)\}$ is bounded away from $\mathbb{S}_i^{n-1}$, we have that the norm of $\mathcal{L}(g_i)(\vec{v}_i^S)/|\mathcal{L}(g_i)(\vec{v}_i^P)|$ is bounded above. As r_i is a repelling fixed point of g_i, we have $\{\mathcal{L}(g_i)(\vec{v}_i^P)\} \to 0$. Since $\{\mathcal{L}(g_i)(\vec{v}_i^P)\} \to 0$, it follows that $\{\mathcal{L}(g_i)(\vec{v}_i^S)\} \to 0$. Hence, $\{\mathcal{L}(g_i)(\vec{v}_i)\} \to 0$.

This implies $\{||\Phi_{t_i}(\mathbf{v})||\} \to 0$ since for the fundamental domain F, the Euclidean metric and the Riemannian metric of $\tilde{\mathbb{V}}_-$ are related by a bounded constant. This is a contradiction. □

A.2. **Neutralized section.** A section $s : U\Sigma \to \mathbb{V}$ is *neutralized* if

$$\nabla_\phi s \in \mathbb{V}_0. \tag{63}$$

We denote by $\Gamma(\mathbb{V})$ the space of sections $U\Sigma \to \mathbb{V}$ and by $\Gamma(\mathbb{V})$ the space of sections $U\Sigma \to \mathbb{V}$.

Recall from [32] the one parameter-group of bounded operators $D\Phi_{t,*}$ on $\Gamma(\mathbb{V})$ and $\Phi_{t,*}$ on $\Gamma(\mathbb{V})$. We denote by ϕ the vector field generated by this flow on $U\Sigma$. Recall Lemma 8.3 of [32] also

Lemma A.4. *If* $\psi \in \Gamma(\mathbb{V})$, *and*

$$t \mapsto D\Phi_{t,*}(\psi)$$

is a path in $\Gamma(\mathbb{V})$ *that is differentiable at* $t = 0$*, then*

$$\frac{d}{dt}|_{t=0}(D\Phi_t)_*(\psi) = \nabla_\phi^{\mathbb{V}}(\psi).$$

Recall that $U\Sigma$ is a recurrent set under the geodesic flow.

Lemma A.5. *A neutralized section exists on* $U\Sigma$*. This lifts to a map* $\tilde{s}_0 : U\Omega \to \mathbb{V}$ *so that* $\tilde{s}_0 \circ \gamma = \gamma \circ \tilde{s}_0$*.*

Proof. Let s a continuous section $U\Sigma \to \mathbb{V}$. We decompose

$$\nabla_\phi(s) = \nabla_\phi^{\mathbb{V}+}(s) + \nabla_\phi^{\mathbb{V}0}(s) + \nabla_\phi^{\mathbb{V}-}(s) \in \mathbb{V}$$

so that $\nabla_\phi^{\mathbb{V}\pm}(s) \in \mathbb{V}_\pm$ and $\nabla_\phi^{\mathbb{V}0}(s) \in \mathbb{V}_0$ hold. Again

$$s_0 = s + \int_0^\infty (D\Phi_t)_*(\nabla_\phi^{\mathbb{V}-}(s))dt - \int_0^\infty (D\Phi_{-t})_*(\nabla_\phi^{\mathbb{V}+}(s))dt$$

is a continuous section and $\nabla_\phi^{\mathbb{V}}(s_0) = \nabla_\phi^{\mathbb{V}0}(s_0)$ as shown in [32].

Since $U\Sigma$ is connected, there exists a fundamental domain F so that we can lift s_0 to $\tilde{s}_0'$ defined on F mapping to $\mathbb{V}$. We can extend $\tilde{s}_0'$ to $U\Omega \to \Omega \times E$. □

Let $N_2(A^n)$ denote the space of codimension two affine spaces of A^n. We denote by $G(\Omega)$ the space of maximal oriented geodesics in Ω. We use the quotient topology on both spaces. There exists a natural action of Γ on both spaces.

For each element $g \in \Gamma - \{\mathrm{I}\}$, we define $N_2(g)$: Now, g acts on ∂A^n with invariant subspaces corresponding to invariant subspace of the linear part $\mathcal{L}(g)$ of g. Since g and g^{-1} are positive proximal, there exists a unique fixed point corresponding to the largest norm eigenvector, an attracting fixed point in ∂A^n, and there exists a unique fixed point corresponding to the smallest norm eigenvector, a repelling fixed point. There exists a $\mathcal{L}(g)$-invariant vector subspace V_g^0 complementary to the join of the subspace generated by these eigenvectors. (This space equals V_0 for the unit tangent vector tangent to the unique maximal geodesic l_g in Ω where g acts on.) It corresponds to a g-invariant subspace $M(g)$ of codimension two in ∂A^n.

Let $\tilde{c}$ be the geodesic in $U\Sigma$ that is g-invariant for $g \in \Gamma$. $\tilde{s}_0(\tilde{c})$ lies on a fixed affine space parallel to V_0^g by the neutrality. There exists a unique affine subspace $N_2(g)$ of codimension two in A^n whose containing $\tilde{s}_0(\tilde{c})$. One immediate property is $N_2(g) = N_2(g^{-1})$.

Definition A.6. We define $S'(\mathrm{bd}\Omega)$ the space of $(n-1)$-dimensional hemispheres with interiors in A^n each of whose boundary in ∂A^n is a supporting hypersphere in ∂A^n to Ω. We denote by $S(\mathrm{bd}\Omega)$ the space of pairs (x, H) where $H \in S'(\mathrm{bd}\Omega)$ and x is in the boundary of H and in $\mathrm{bd}\Omega$.

Define Δ to be the diagonal set of $\mathrm{bd}\Omega \times \mathrm{bd}\Omega$. Denote by $\Lambda^* = \mathrm{bd}\Omega \times \mathrm{bd}\Omega - \Delta$. Let $G(\Omega)$ denote the space of maximal oriented geodesics in Ω. $G(\Omega)$ is in one-to-one correspondence with Λ^* by the map taking the maximal oriented geodesic to the ordered pair of its endpoints.

Proposition A.7.

- *There exists a continuous function* $\hat{s} : U\Omega \to N_2(A^n)$ *equivariant with respect to* Γ*-actions.*

- *Given* $g \in \Gamma$ *and for the unique unit speed geodesic* $\vec{l}_g$ *in* $U\Omega$ *lying over a geodesic* l_g *where* g *acts on,* $\hat{s}(\vec{l}_g) = \{N_2(g)\}$.
- *This gives a continuous map*

$$\tau : \mathrm{bd}\Omega \times \mathrm{bd}\Omega - \Delta \to N_2(A^n)$$

again equivariant with respect to the Γ*-actions. There exists a continuous function*

$$\tau : \Lambda^* \to S(\mathrm{bd}\Omega).$$

Proof. Given a vector $\vec{u} \in U\Omega$, we find $\tilde{s}_0(\vec{u})$. There exists a lift $\tilde{\phi}_t : U\Omega \to U\Omega$ of the geodesic flow ϕ_t. Then $\tilde{s}_0(\tilde{\phi}_t(\vec{u}))$ all lies in an affine subspace H^{n-2} parallel to V_0 for $\vec{u}$ by the neutrality condition equation 63. We define $\hat{s}(\vec{u})$ to be this H^{n-2}.

We see that for any unit vector $\vec{u}'$ on the maximal (oriented) geodesic in Ω determined by $\vec{u}$, we obtain that $\hat{s}(\vec{u}')$ equals H^{n-2}. Hence, this determines the continuous map $\bar{s} : G(\Omega) \to N_2(A^n)$. The Γ-equivariance comes from that of $\tilde{s}_0$.

For $g \in \Gamma$, $\vec{u}$ and $g(\vec{u})$ lie on the g-invariant geodesic l_g provided $\vec{u}$ is tangent to l_g. Since $g(\tilde{s}_0(\vec{u})) = \tilde{s}_0(g(\vec{u}))$ by equivariance, $g(\tilde{s}_0(\vec{u}))$ lies on $\hat{s}(\vec{u}) = \hat{s}(g(\vec{u}))$ by two paragraphs above and $g(\bar{s}(l_g)) = \bar{s}(l_g)$.

The third item follows since $\mathrm{bd}\Omega \times \mathrm{bd}\Omega - \Delta$ is in one-to-one correspondence with the space $G(\Omega)$.

The last item follows by taking for each pair $(x, y) \in \Lambda^*$ we take the geodesic l with endpoints x and y, and taking the hyperspace in A^n containing $\bar{s}(l)$ and its boundary containing x. □

We define $h(x, y)$ by letting $\tau(x, y) = (x, h(x, y))$. Recall the properly convex open Γ-invariant domain U with boundary in $\mathrm{Cl}(\Omega)$.

A.3. **Asymtotic niceness.**

Lemma A.8. *Suppose that* x *and* y *are fixed points of an element* g *of* Γ *in* $\mathrm{bd}\Omega$. *Then* $h(x, y)$ *is disjoint from* U.

Proof. Suppose not. $h(x, y)$ is a g-invariant hemisphere, and x is a fixed point of g in it, Then $U \cap h(x, y)$ is a g-invariant properly convex open domain containing x in its boundary.

Suppose first that $h(x, y)$ has a fixed point z of g. Then the associated eigenvalue is strictly less than that of x by the uniform middle-eigenvalue condition and hence z is in the closure of $U \cap h(x, y)$. We form the 2-sphere P containing x, y, z invariant under g. Then considering the action of g on $P \cap U$, we easily obtain that $P \cap U$ cannot be properly convex due to the fact that z is a saddle-type fixed point. Hence, there exists no fixed point z.

Now, we consider the only alternative that $h(x, y)$ contains a g-invariant affine subspace A' of codimension 2. $g|h(x, y)$ has the largest norm eigenvalue at x, x_-. Therefore, acting by $\langle g \rangle$ on a generic point z of $h(x, y) \cap U$ gives us an arc in $h(x, y)$ with endpoints x or x_- and an endpoint y' in $\mathrm{bd}A' \subset \mathrm{bd}A^n$. Where y' is a fixed point in $h(x, y)$ different from y as $y \notin h(x, y)$. Since $x \in \mathrm{Cl}(\Omega)$, it follows that x_- is not in $\mathrm{Cl}(\Omega)$ for the proper convexity. Since $x, y' \in \mathrm{Cl}(\Omega)$, we have $\overline{xy'} \subset \partial A^n$ is a subset of $\mathrm{Cl}(\Omega)$. Also,

we have $\overline{xy'} \subset \partial h(x, y)$ where $\partial h(x, y)$ is a supporting subspace of $\mathsf{Cl}(\Omega)$. However, the strict convexity of Ω implies that we cannot have a segment in $\mathsf{bd}\Omega$. (See Benoist [1].) $\square$

The proof of the following lemma is the different from one in [20].

Lemma A.9. *Let $(x, y) \in \Lambda^*$. Then $\tau(x, y)$ does not depend on y, and $h(x, y)$ is never a hemisphere in ∂A^n for every $(x, y) \in \Lambda^*$. Hence, we obtain a continuous function $\tau : \mathsf{bd}\Omega \to S(\mathsf{bd}\Omega)$.*

Proof. We claim that for any x, y in $\mathsf{bd}\Omega$, $h(x, y)$ is disjoint from U: Since the fixed points in $\mathsf{bd}\Omega$ are doubly dense, we can find a sequence $x_i \to x$ and $y_i \to y$ where x_i and y_i are fixed points of an element $g_i \in \Gamma$ for each i. If $h(x, y) \cap U \neq \emptyset$, then $h(x_i, y_i) \cap U \neq \emptyset$ for i sufficiently large by the continuity of the map τ. This is a contradiction by Lemma A.8

Notice also $h(x, y)$ is never in ∂A as it is disjoint from U.

Let $H(x, y)$ denote the half-space bounded by $h(x, y)$ containing U. For each x, we define

$$H(x) := \bigcap_{y \in \mathsf{bd}\Omega - \{x\}} H(x, y).$$

Define $h(x)$ as the boundary $(n-1)$-hemisphere of $H(x)$. (Note that $\partial H(x, y')$ is supporting $\mathsf{bd}\Omega$ and hence is independent of y' as $\mathsf{bd}\Omega$ is C^1.)

Let $H(x)$ denote the open half-space bounded by $h(x)$ containing U. Let U' be defined as the convex open domain $\bigcap_{x \in \mathsf{bd}\Omega} H(x)$ containing U. Since $\mathsf{bd}\Omega$ has at least $n+1$ points in general position and tangent hemispheres, U' is properly convex. Let U'' be the open domain $\bigcap_{x \in \mathsf{bd}\Omega}(E - \mathsf{Cl}(H(x)))$. It has the boundary $\mathcal{A}(\mathsf{Cl}(\Omega))$ for the antipodal map $\mathcal{A}$ and is a properly convex domain as the antipodal set of $\mathsf{bd}\Omega$ has at least $n+1$ points in general position. Note that $U' \cap U'' = \emptyset$.

If for some x, y, $h(x, y)$ is different from $h(x)$, then $h(x, y) \cap U'' \neq \emptyset$. This is a contradiction as the proof of Lemma A.8. Thus, it follows that $h(x, y) = h(x)$ for all $y \in \mathsf{bd}\Omega - \{x\}$.

We show the continuity of $x \mapsto h(x)$: Let $x_i \in \mathsf{bd}\Omega$ be a sequence converging to $x \in \mathsf{bd}\Omega$. Then choose $y_i \in \mathsf{bd}\Omega$ so that $y_i \to y$ and we have $h(x_i) = h(x_i, y_i)$ converges to $h(x, y) = h(x)$ by the continuity of τ. Therefore, h is continuous. $\square$

Proof of Theorem A.1. For each point $x \in \mathsf{bd}\Omega$, we obtain an $(n-1)$-dimensional hemisphere $h(x)$ passing E with $\partial h(x) \subset \partial A^n$ supporting Ω by Lemma A.9. There exists a half-space $H(x) \subset E$ bounded by $h(x)$ and containing Ω.

We form the properly convex open domain $\bigcap_{x \in \mathsf{bd}\Omega} H(x)$ containing U. The uniqueness of $h(x)$ in the proof of Lemma A.9 gives us the unique asymptotic totally geodesic hypersurfaces.

$\square$

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